Neutrosophic

Triplet

Structures

Volume I

pons

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Neutrosophic Science International Association

NEUTROSOPHIC TRIPLET STRUCTURES

Volume I

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Neutrosophic Triplet Structures

Volume I

Aims and Scope

Neutrosophic theory and its applications have been expanding in all directions at an astonishing rate especially after of the introduction the journal entitled "Neutrosophic Sets and Systems". New theories, techniques, algorithms have been rapidly developed. One of the most striking trends in the neutrosophic theory is the hybridization of neutrosophic set with other potential sets such as rough set, bipolar set, soft set, hesitant fuzzy set, etc. The different hybrid structures such as rough neutrosophic set, single valued neutrosophic rough set, bipolar neutrosophic set, single valued neutrosophic hesitant fuzzy set, etc. are proposed in the literature in a short period of time. Neutrosophic set has been an important tool in applications to various areas such as data mining, decision making, e-learning, engineering, medicine, social science, and some more.

Florentin Smarandache, Memet Şahin

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Preface

Neutrosophic set has been derived from a new branch of philosophy, namely Neutrosophy. Neutrosophic set is capable of dealing with uncertainty, indeterminacy and inconsistent information. Neutrosophic set approaches are suitable to modeling problems with uncertainty, indeterminacy and inconsistent information in which human knowledge is necessary, and human evaluation is needed.

Neutrosophic set theory was firstly proposed in 1998 by Florentin Smarandache, who also developed the concept of single valued neutrosophic set, oriented towards real world scientific and engineering applications. Since then, the single valued neutrosophic set theory has been extensively studied in books and monographs, the properties of neutrosophic sets and their applications, by many authors around the world. Also, an international journal - Neutrosophic Sets and Systems started its journey in 2013.

Neutrosophic triplet was first defined in 2016 by Florentin Smarandache and Mumtaz Ali and they also introduced the neutrosophic triplet groups in the same year. For every element "x" in a neutrosophic triplet set A, there exist a neutral of "x" and an opposite of "x". Also, neutral of "x" must be different from the classical neutral element. Therefore, the NT set is different from the classical set. Furthermore, a NT of "x" is showed by
<x, neut(x), anti(x)>.

This first volume collects original research and applications from different perspectives covering different areas of neutrosophic studies, such as decision making, Triplet, topology, and some theoretical papers.

This volume contains three sections: NEUTROSOPHIC TRIPLET, DECISION MAKING, AND OTHER PAPERS.

SECTION ONE

Neutrosophic Triplet Research

Chapter One

Neutrosophic Triplet Partial Inner Product Space

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Abstract

In this chapter, we obtain neutrosophic triplet partial inner product space. We give some definitions and examples for neutrosophic triplet partial inner product space. Then, we obtain some properties and we prove these properties. Furthermore, we show that neutrosophic triplet partial inner product space is different from neutrosophic triplet inner product space and classical inner product space.

Keywords: neutrosophic triplet partial inner product space, neutrosophic triplet vector spaces, neutrosophic triplet partial normed spaces, neutrosophic triplet partial metric spaces

1. Introduction

Smarandache introduced neutrosophy in 1980, which studies a lot of scientific fields. In neutrosophy [1], there are neutrosophic logic, set and probability. Neutrosophic logic is a generalization of a lot of logics such as fuzzy logic [2] and intuitionistic fuzzy logic [3]. Neutrosophic set is denoted by (t, i, f) such that "t" is degree of membership, "i" is degree of indeterminacy and "f" is degree of non-membership. Also, a lot of researchers have studied neutrosophic sets [4-9,24-28]. Furthermore, Smarandache *et al.* obtained neutrosophic triplet (NT) [10] and they introduced NT groups [11]. For every element "x" in neutrosophic triplet set A, there exist a neutral of "a" and an opposite of "a". Also, neutral of "x" must different from the classical unitary element. Therefore, the NT set is different from the classical set. Furthermore, a NT "x" is denoted by by <x, neut(x), anti(x)>. Also, many researchers have introduced NT structures [12-20]

Inner product is a special operator (<. , .>) built on vector spaces and it has certain properties. Also, if (<. , .>) is an inner product on a vector space, the vector space is called inner product space. The Hilbert space (every Cauchy sequence is convergent in it) is a special inner product space and it also has certain properties. In functional analysis, inner product space and Hilbert space are a broad topic with wide area of applications. Also, recently many researchers have introduced inner product space [21-23].

In this chapter, we obtain NT inner product space. In section 2; we give definitions of NT set [11], NT field [12], NT partial metric space [17], NT vector space [13] and NT partial normed space [20]. In section 3, we introduce NT partial inner product space and give some properties and examples for NT partial inner product space. Also, we show that NT partial inner product space is different from the NT inner product space and classical inner product space. Furthermore, we show relationship between NT partial metric spaces, NT partial normed space with NT partial inner product space. Also, we give definition of convergence of sequence, Cauchy sequence and Hilbert space in NT inner product space. In section 4, we give conclusions.

2.Basic and fundamental concepts

Definition 2.1: [11] Let * be a binary operation. (X, #) is a NT set (NTS) such that

i) There must be neutral of "x" such x#neut(x) = neut(x)#x = x, x ∈ X.

ii) There must be anti of "x" such x#anti(x) = anti(x)#x = neut(x), x ∈ X.

Furthermore, a NT "x" is showed with (x, neut(x), anti(x)).

Also, neut(x) must different from classical unitary element.

Definition 2.2: [12] Let (X, &, $) be a NTS with two binary operations & and $. Then (X, &, $) is called NT field (NTF) such that

1. (F, &) is a commutative NT group,

2. (F, $) is a NT group

3. x$ (y&z)= (x$y) & (x$z) and (y&z)$x = (y$x) & (z$x) forv every x, y, z ∈ X.

Definition 2.3: [17] Let (A, #) be a NTS and m#n ∈ A, ∀ m, n ∈ A. NT partial metric (NTPM) is a map p_N: A x A → $\mathbb{R}^+ \cup \{0\}$ such that ∀ m, n, k ∈ A

i) p_N(m, n) ≥ p_N(n, n)≥0

ii) If p_N(m, m) = p_N(m, n) = p_N(n, n) = 0, then there exits at least one m, n pair such that m = n.

iii) p_N(m, n) = p_N(n, m)

iv) If there exists at least an element n∈A for each m, k∈ A pair such that

p_N(m, k)≤ p_N(m, k#neut(n)), then p_N(m, k#neut(n))≤ p_N(m, n) + p_N(n, k) - p_N(n, n)

Also, ((A, #), p_N) is called NTPM space (NTPMS).

Definition 2.4: [13] Let $(F, \&_1, \$_1)$ be a NTF and let $(V, \&_2, \$_2)$ be a NTS with binary operations "$\&_2$" and "$\$_2$". If $(V, \&_2, \$_2)$ is satisfied the following conditions, then it is called a NT vector space (NTVS),

1) $x\&_2 y \in V$ and $x \$_2 y \in V$; for every x, y $\in$ V

2) $(x\&_2 y) \&_2 z = x\&_2 (y\&_2 z)$; for every x, y, z $\in$ V

3) $x\&_2 y = y\&_2 x$; for every x, y $\in$ V

4) $(x\&_2 y) \$_2 m = (x\$_2 m) \&_2 (y\$_2 m)$; for every m$\in$ F and every x, y $\in$ V

5) $(m\&_1 n) \$_2 x = (m\$_2 x) \&_1 (n\$_2 x)$; for every m, n $\in$ F and every u $\in$ V

6) $(m\$_1 n) \$_2 x = m\$_1 (n\$_2 x)$; for every m, n $\in$ F and every x $\in$ V

7) For every x $\in$ V, there exists at least a neut(y) $\in$ F such that

$x \$_2 neut(y) = neut(y) \$_2 x = x$

Definition 2.5: [20] Let $(V, *_2, \#_2)$ be NTVS on $(F, *_1, \#_1)$ NTF. $\|.\|: V \to \mathbb{R}^+ \cup \{0\}$ is a map that it is called NT partial norm (NTPN) such that

a) f: F X V $\to \mathbb{R}^+ \cup \{0\}$ is a function such that f(α,x) = f(anti(α), anti(x)); for all a, b $\in$ V and m $\in$ F;

b) $\|a\| \geq 0$;

c) If $\|a\| = \|neut(a)\| = 0$, then a = neut(a)

d) $\|m\#_2 a\| = f(m,a).\|a\|$

e) $\|anti(a)\| = \|a\|$

f) If there exists at least k element for each a, b pair such that

$\|\|a*_2 a\| + \|neut(k)\| \leq \|a*_2 b*_2 neut(k)\|$; then $\|a*_2 b\| \leq \|a\| + \|b\| - \|neut(k)\|$, for any k $\in$ V.

Furthermore, $((NTV, *_2, \#_2), \|.\|)$ is called NTPN space (NTPNS).

Theorem 2.6: [20] Let $((N, *_2, \#_2), \|.\|)$ be a NTPNS on $(F, *_1, \#_1)$ NTF. Then, the function is p: V x V$\to \mathbb{R}$, p(x, y) = $\|x*_2$ anti(y)$\|$ is a NTPMS.

Definition 2.7: [14] Let $(V, *_2, \#_2)$ be a NTVS on $(F, *_1, \#_1)$ NTF. Then, $<.,.>: V \times V \to \mathbb{R}^+ \cup \{0\}$ is a NT inner product (NTIP) on $(V, *_2, \#_2)$ such that

a) f: F X V X V $\rightarrow \mathbb{R}^+ \cup \{0\}$ is a function such that f(α, a, b)= f(anti(α), anti(a), anti(a)), for all a, b, c $\in$ NTV and m, n $\in$ F,

 b) <a, a> $\geq$ 0;

 c) If a= neut(a), then <a, a> =0

 d) <(m$\#_2$ a)$*_2$ (n$\#_2$ b), c> = f(m, a, c). <x, c>+ f(n,b,z). <b, c>

 e) <anti(a), anti(a)> = <a, a>

 f) <a, b>=<b, a>

Also, ((NTV, $*_2$, $\#_2$), <., .>) is called NTIP space (NTIPS).

Theorem 2.8: [14] Let (V, $*_2$, $\#_2$) be a NTVS on (F, $*_1$, $\#_1$) NTF and let ((V, $*_2$, $\#_2$), <. , .>) be a NTIPS on (V,$*_2$, $\#_2$) and f: F X V X V $\rightarrow \mathbb{R}^+ \cup \{0\}$ be a map such that f(α, a, b)= f(anti(α), anti(a), anti(b)) for all a, b $\in$ V and m, n $\in$ F. Then,

 <(m$\#_2$ a)$*_2$ (n$\#_2$ b), (m$\#_2$ a)$*_2$ (n$\#_2$ b)> =

 f(m, (m$\#_2$ a)$*_2$ (n$\#_2$ b), a).f(m, a, a).<a, a> +

 [f(m, (m$\#_2$a)$*_2$ (n$\#_2$ b), a). f(n, a, b) + f(n, (m$\#_2$ a)$*_2$ ($n\#_2$ b), b).f(m, a, b)].<a, b> +

 f(n, (m$\#_2$ a)$*_2$ (n$\#_2$ b), b). f(n, b, b).<b, b>.

3. Neutrosophic Triplet Partial Inner Product Space

Definition 3.1: Let (V, $*_2$, $\#_2$) be NTVS on (F, $*_1$, $\#_1$) NTF. Then, <., .> :V x V $\rightarrow \mathbb{R}^+ \cup \{0\}$ is a NT partial inner product (NTPIP) on (V,$*_2$, $\#_2$) such that

 i) f: F X V X V $\rightarrow \mathbb{R}^+ \cup \{0\}$ is a function such that f(m, a, b)= f(anti(m), anti(a), anti(b)) for all a, b, c $\in$ V and m, n $\in$ F;

 ii) <a, b> $\geq$ 0 and <a, a> $\geq$ 0;

 iii) If <a, a> = <neut(a), neut(a)> = 0, then a = neut(a)

 iv) <(m$\#_2$ a)$*_2$ (n$\#_2$ b), c> = f(m, a, c). <a, c>+ f(n,b,c). <b, c>

 v) <anti(a), anti(a)> = <a, a>

 vi) <a, b> = <b, a>

        Also, ((NTV, $*_2$, $\#_2$), <., .>) is called NTPIP space (NTPIPS).

Corollary 3.2: By Definition of NTPIPS that NTPIPS is different from the classical inner product spaces, since conditions i) and ii) are different in classical inner product space.

Corollary 3.3: In Definition 2.7, if x = neut(x) then <x, x> = 0. In Definition 3.1, <x, x> = <neut(x), neut(x)> = 0 then x = neut(x). Thus, NTPIPS is different from NTIPS.

Corollary 3.4:

i) For a NTPIPS, if a = neut(a) and <a, a> = 0, then NTPIPS is a NTIPS.

ii) For a NTIPS, if   <a, a> = <neut(a), neut(a)> = 0 and a = neut(a), then NTIPS is a NTPIPS.

Example 3.5: Let X = {∅ , {1}, {2}, {1, 2}}. From Definition 2.4, (X, ∪, ∩) is a NTVS on the   (X, ∪, ∩) NTF. Also,

The NTs with respect to ∪;

    neut(K) = K and anti(K) = K,

The NTs with respect to ∩;

    neut(M) = M and anti(M) = M,

    Now, we take <. , .>:X x X → $\mathbb{R}^+\cup\{0\}$ such that <K, L> = $s(K) + s(L)$ and s(K) is number of elements in K ∈ X and K' is complement of K ∈ X.

 We show that <. , .> is a NTPIP and ((X, ∪, ∩), <. , .>) is a NTPIPS.

i) We can take f: X × X×X → $\mathbb{R}^+\cup\{0\}$ such that

$$f(A,B,C) = \begin{cases} \frac{s(C)+2.s(A\cap B)}{2(s(B)+s(C))}; & if \quad A\cap B\cap C = \emptyset, B = C \neq \emptyset \\ \frac{s(C)-s(A)+2.s(A\cap B)}{2(s(B)+s(C))}; & if \quad A\cap B\cap C \neq \emptyset, B = C \neq \emptyset \\ 0; & if \quad B = C = \emptyset \\ \frac{1}{2(s(B)+s(C))}; & if\, A\cap B\cap C = \emptyset, B = C \neq \emptyset, s(C)+s(A\cap B) = 0, s(C)-s(A)+2.s(A\cap B) \end{cases}$$

Also, f(A,B,C) = f(anti(A), anti(B), anti(C)) because anti(A) = A, for all A ∈ X.

ii) <A, B> = $s(A) + s(B)$≥ 0 and <A, A> = $s(A) + s(A)$≥ 0

iii) For A = ∅, $s(A) + s(A) = s(neut(A)) + s(neut(A)) = 0$, Also, ∅ = neut(∅)

iv) For A = ∅, B = {1}, C = {2}, D = {1, 2}, E = ∅;

<((∅∩{1})∪ ({2} ∩ {1, 2}), ∅> = $s((\emptyset \cap \{1\}) \cup (\{2\} \cap \{1, 2\}))$ + s(∅) = 1. Also,

f(∅, {1}, ∅) = 1 /2,

<{1}, ∅> = 1,

f({2}, {1,2}, ∅) = 1 / 4,

<{1,2}, ∅> = 2.

Thus, <(∅∩{1})∪ ({2} ∩ {1, 2}), ∅> =

f(∅, {1}, ∅).<{1}, ∅> + f({2}, {1,2}, ∅).<{1,2}, ∅>.

For A = ∅, B = {1}, C = {2}, D = {1, 2} , E = {1};

<((∅∩{1})∪ ({2} ∩ {1, 2}), {1}> = s((∅ ∩ {1}) ∪ ({2} ∩ {1, 2})) + s({1}) = 2. Also,

f(∅, {1}, {1}) = 1 /2

<{1}, {1}> = 2

f({2}, {1,2}, {1}) = 1 / 3

<{1,2}, {1}> = 3.

Thus, <(∅∩{1})∪ ({2} ∩ {1, 2}), {1}> =

f(∅, {1}, {1}).<{1}, {1}> + f({2}, {1,2}, {1}).<{1,2}, {1}>.

For A = ∅, B = {1}, C = {2}, D = {1, 2} , E = {2};

<((∅∩{1})∪ ({2} ∩ {1, 2}), {2}> = s((∅ ∩ {1}) ∪ ({2} ∩ {1, 2})) + s({2}) = 2. Also,

f(∅, {1}, {2}) = 1 /2

<{1}, {2}> = 2

f({2}, {1,2}, {2}) = 1 / 3

<{1,2}, {2}> = 3.

Thus, <(∅∩{1})∪ ({2} ∩ {1, 2}), {2}> =

 f(∅, {1}, {2}).<{1}, {2}> + f({2}, {1,2}, {2}).<{1,2}, {2}>.

For A = ∅, B = {1}, C = {2}, D = {1, 2}, E = {1, 2};

<((∅∩{1})∪ ({2} ∩ {1, 2}), {1, 2}> = s((∅ ∩ {1}) ∪ ({2} ∩ {1, 2})) + s({1, 2}) = 3. Also,

f(∅, {1}, {1, 2}) = 2 /3

$<\{1\}, \{1,2\}> = 3$

$f(\{2\}, \{1,2\}, \{1,2\}) = 2 / 8 = 1/4$

$<\{1,2\}, \{1, 2\}> = 4.$

Thus, $<(\emptyset \cap \{1\}) \cup (\{2\} \cap \{1, 2\}), \{1,2\}> = f(\emptyset, \{1\}, \{1,2\}).<\{1\}, \{1,2\}> +$

$f(\{2\}, \{1,2\}, \{1,2\}).<\{1,2\}, \{1,2\}>.$

  Furthermore, condition iv is satisfies by another A, B, C, D, E elements.

v) $<\text{anti}(K), \text{anti}(K)> = s(\text{anti}(K)) + s(\text{anti}(K)) = s(K) + s(K) = <K, K>$ because $K = \text{anti}(K)$, for every $K \in X$.

vi) $<K, L> = s(K) + s(L) = s(L) + s(K) = <L, K>.$

Theorem 3.6: Let $(V, *_2, \#_2)$ be a NTVS on $(F, *_1, \#_1)$ NTF and let $((V, *_2, \#_2), <., .>)$ be a NTIPS on $(V, *_2, \#_2)$ and $f: F \times V \times V \to \mathbb{R}^+ \cup \{0\}$. For every $a, b, c \in V$ and $m, n \in F$, if $a \neq b$, $a = c$ or $b = c$ and $<c, c> \geq 1$ and $<c, c> \leq <a, a>, <b, b>$, then

$$(< a, b >)^2 \leq <a, a>.<b, b> - <c, c>$$

Proof: We suppose $a \neq b$, $a = c$ or $b = c$. From the Theorem 2.6; if we take

$$f(m, (m\#_2 a) *_2 (n\#_2 b), a,) = \frac{<a,b>^2 + <c,c>^{1/2}}{<a,a>^{3/2}}$$

$$f(m, a, b) = \frac{-(<a,b> + \frac{<c,c>^{1/2}}{<a,b>})}{<a,a>^{1/2}}$$

$$f(n, b, b) = \frac{1}{<b,b>^{1/2}}$$

$$f(n, a, b) = \frac{(1 + \frac{<c,c>^{1/2}}{<a,b>})}{<a,a>^{1/2}}$$

$f(n, (m\#_2 a) *_2 (n\#_2 b), a) = f(m, a, a) = f(n, (m\#_2 a) *_2 (n\#_2 b), b) = 1$, then

$<(m\#_2 a) *_2 (n\#_2 b), (m\#_2 a) *_2 (n\#_2 b)> =$

$f(m, (m\#_2 a) *_2 (n\#_2 b), a).f(m, a, a).<a, a> +$

$[f(m, (m\#_2 a) *_2 (n\#_2 b), a). f(n, a, b) + f(n, (m\#_2 a) *_2 (n\#_2 b), b).f(m, a, b)].<a, b> +$

$f(n, (m\#_2 a) *_2 (n\#_2 b), b). f(n, b, b).<b, b> =$

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$$\frac{<a,b>^2+<c,c>^{1/2}}{<a,a>^{3/2}} \cdot <a,a>- \frac{(<a,b>+\frac{<c,c>^{1/2}}{<a,b>})}{<a,a>^{1/2}} \cdot <a,b>- \frac{(1+\frac{<c,c>^{1/2}}{<a,b>})}{<a,a>^{1/2}} \cdot <a,b>+ \frac{1}{<b,b>^{1/2}} <b,b> =$$

$$- \frac{(1+\frac{<c,c>^{1/2}}{<a,b>})}{<a,a>^{1/2}} \cdot <a,b>+ <b,b>^{1/2} = <b,b>^{1/2} - \frac{<a,b>+<c,c>^{1/2}}{<a,a>^{1/2}}. \text{ Thus, we have}$$

$$<b,b>^{1/2} - \frac{<a,b>+<c,c>^{1/2}}{<a,a>^{1/2}} \geq 0 \text{ and}$$

$$<a,a>^{1/2} \cdot <b,b>^{1/2} - <a,a>^{1/2} \cdot \frac{<a,b>+<c\ c>^{1/2}}{<a,a>^{1/2}} =$$

$$<a,a>^{1/2} \cdot <b,b>^{1/2} - <a,b> - <c,c>^{1/2} \geq 0. \text{ Thus,}$$

$$<a,b> \leq <a,a>^{1/2} \cdot <b,b>^{1/2} - <c,c>^{1/2}.$$

Theorem 3.7: Let $(V,*_2, \#_2)$ be a NTVS on $(F,*_1, \#_1)$ NTF and let $((V, *_2, \#_2), <.\,,\,.>)$ be a NTPIPS on $(V,*_2, \#_2)$. For every $a,b,c \in V$ and $m,n \in F$, if $f(m, a, a) = f(m, a)$ and $\|a\| = \sqrt{<a,a>}$. Then, $((V, *_2, \#_2), \|.\|)$ is a NTPMS on $(V,*_2, \#_2)$.

Proof: We show that $\|a\| = \sqrt{<a,a>}$ is a NTPMS. From Definition 3.1 and Definition 2.5,

a) $\|a\| = \sqrt{<a,a>} \geq 0$.

b) If $\|a\| = \sqrt{<a,a>} = \sqrt{<neut(a),neut(a)>} = \|neut(a)\| = 0$, then $a = neut(a)$

c) $\|m\#_2 a\| = \sqrt{<m\#_2 a, m\#_2 a>} = \sqrt{f(m,a,a)}.\sqrt{f(m,a,a)} \cdot \sqrt{<a,a>} = f(m,a,a).\sqrt{<a,a>}=$
   $f(m,a).\|a\|$

d) $\|anti(x)\| = \sqrt{<anti(x),anti(x)>} = \sqrt{<a,a>} == \|a\|$

e) In Theorem 2.6; if we take $m= neut(a)$, $n = neut(b)$, then
   $<(m\#_2 a)*_2 (n\#_2 b), (m\#_2 a)*_2 (n\#_2 b)> = <a *_2 b, a *_2 a> =$

   $f(neut(a), a*_2 b, a).f(neut(a), a, a).<a, a> +$

   $[f(neut(a), a*_2 b, a).f(neut(b), a, b)+ f(neut(b), a*_2 b, b).f(neut(a), a, b)].<a, b> +$

   $f(neut(b), a*_2 b, b).f(neut(b), b, b).<b, b>$.
   Also, if we take
   $f(neut(a), a*_2 b, a) = f(neut(a), a, a) = f(neut(b), a, b) = f(neut(b), a*_2 b, b) = f(neut(a), a, b) = f(neut(b), b, b) = 1$, then
   $<a *_2 b, a *_2 b> = <a, a> + 2.<a, b> + <b, b> = \|a\|^2 + 2.<a,b> + \|a\|^2$. From the Theorem 3.6, if $a \neq b$, $a = c$ or $b = c$, then

   $$<a,b> \leq \sqrt{<a,a>}.\sqrt{<b,b>} - \sqrt{<c,c>}.\text{Thus,}$$

$$< a *_2 b , a *_2 b >= \|a\|^2 + 2.<a, b>+ \|b\|^2 \leq$$

$$\|a\|^2 + 2\|a\|\|b\| + \|b\|^2 - 2\|b\| =$$

$$(\|a\| + \|b\|)^2 - 2\|b\| \tag{1}$$

If we take neut(c) = neut(b), then

$$< a *_2 b *_2 neut(c)\ ,\ a *_2 b *_2 neut(c)> = < a *_2 b, a *_2 b >$$

Therefore, if we take a = b, it is clear that

$$< a *_2 b, a *_2 b > - < neut(m), neut(m) >^{1/2} \leq$$

$$< a *_2 b *_2 neut(c)\ ,\ a *_2 b *_2 neut(c)> \tag{2}$$

Furthermore, from Definition 2.5, If there exists at least k = neut(m) element for a, b such that $\|a *_2 b\| - \|neut(k)\| \leq \|a *_2 b *_2 neut(k)\|$, then $\|a *_2 b\| \leq \|a\| + \|b\| - \|neut(k)\|$. Thus, we can take from (1), (2)

$$\|a *_2 b\| \leq \|a\| + \|b\| - \|neut(k)\|.$$

Corollary 3.8: Every NTPMS is reduced by a NTPIPS. But the opposite is not always true. Similarly; every NTPNS is reduced by a NTPIPS. But the opposite is not always true.

Definition 3.9: $((X, \&_2 , \$_2), <. , .>)$ be a NTPIPS on $(Y, \&_1 , \$_1)$ NTF and $((X, \&_2, \$_2), <. , .>)$ be a NTPIPS such that $\|a\| = \sqrt{< a, a >}$. Then, p: X x X→ $\mathbb{R}$ is a NTPM define by

$$p(a, b) = \| a *_2 anti(b)\| = \sqrt{< a *_2 \ anti(b), a *_2 \ anti(b) >}$$

and is called NTPM reduced by $((X, \&_2, \$_2), <. , .>)$.

Definition 3.10: $((X, \&_2, \$_2), <. , .>)$ be a NTPIPS on $(Y, \&_1, \$_1)$ NTF, $\{x_n\}$ be a sequence in NTPIPS and p be a NTPM reduced by $((X, \&_2, \$_2), <. , .>)$. For all ε>0, x, k ∈ X

$$p(x, \{x_n\}) = < x *_2 \ anti(\{x_n\}), x *_2 \ anti(\{x_n\}) >^{1/2} < \varepsilon + p(k, k);$$

if there exists a M∈ $\mathbb{N}$ such that for all n ≥M, then $\{x_n\}$ sequence converges to a. It is denoted by

$$\lim_{n \to \infty} x_n = a \text{ or } x_n \to a.$$

Definition 3.11: $((X, \&_2, \$_2), <. , .>)$ be a NTPIPS on $(Y, \&_1, \$_1)$ NTF, $\{x_n\}$ be a sequence in NTPIPS and p be a NTPM reduced by $((X, \&_2, \$_2), <. , .>)$. For all $\varepsilon > 0$, x, k $\in$ X such that for all n $\geq$ M

$p((\{x_m\}, \{x_n\}) =$

$\|x *_2 \text{anti}(\{x_n\})\| < (\{x_m\} *_2 \text{anti}(\{x_n\})), (\{x_m\} *_2 \text{anti}(\{x_n\})) >^{1/2} < \varepsilon + p(k, k);$

if there exists a M $\in \mathbb{N}$; $\{x_n\}$ sequence is called Cauchy sequence.

Definition 3.12: Let $((X, \&_2, \$_2), <. , .>)$ be a NTPIPS on $(Y, \&_1, \$_1)$ NTF, $\{x_n\}$ be a sequence in this space and p be a NTPM reduced by $((X, \&_2, \$_2), <. , .>)$. If each $\{x_n\}$ Cauchy sequence in NTPIPS is convergent by p NTPM reduced by $((X, \&_2, \$_2), <. , .>)$, then $((X, \&_2, \$_2), <. , .>)$ is called Hilbert space in NTPIPS.

Conclusions

In this chapter, we obtained NTPIPS. We also showed that NTPIPS is different from the NTIPS and classical inner product space. Then, we defined Hilbert space for NTPIPS. Thus, we have added a new structure to NT structure and gave rise to a new field or research called NTPIPS. Also, thanks to NTPIPS researcher we obtained new structures and properties.

Abbreviations

NT: Neutrosophic triplet

NTS: Neutrosophic triplet set

NTF: Neutrosophic triplet field

NTVS: Neutrosophic triplet vector space

NTPM: Neutrosophic triplet partial metric

NTPMS: Neutrosophic triplet partial metric

NTPN: Neutrosophic triplet partial norm

NTPNS: neutrosophic triplet partial norm space

NTIPS: neutrosophic triplet inner product space

NTPIP: neutrosophic triplet partial inner product

NTPIPS: neutrosophic triplet partial inner product space

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Chapter Two

Neutrosophic Triplet Partial v-Generalized Metric Space

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Abstract

In this chapter, study the notion of neutrosophic triplet partial v-generalized metric space. Then, we give some definitions and examples for neutrosophic triplet partial v-generalized metric space and obtain some properties and prove these properties. Furthermore, we show that neutrosophic triplet partial v-generalized metric space is different from neutrosophic triplet v-generalized metric space and neutrosophic triplet partial metric space.

Keywords: neutrosophic triplet metric space, neutrosophic triplet partial metric space, neutrosophic triplet v- generalized metric spaces, neutrosophic triplet partial v- generalized metric spaces

1. Introduction

Smarandache introduced neutrosophy in 1980, which studies a lot of scientific fields. In neutrosophy, there are neutrosophic logic, set and probability in [1]. Neutrosophic logic is a generalization of a lot of logics such as fuzzy logic in [2] and intuitionistic fuzzy logic in [3]. Neutrosophic set is denoted by (t, i, f) such that "t" is degree of membership, "i" is degree of indeterminacy and "f" is degree of non-membership. Also, a lot of researchers have studied neutrosophic sets in [4-9, 35-39]. Furthermore, Smarandache and Ali obtained neutrosophic triplet (NT) in [10] and they introduced NT groups in [11]. For every element "x" in neutrosophic triplet set A, there exist a neutral of "a" and an opposite of "a". Also, neutral of "x" must different from the classical unitary element. Therefore, the NT set is different from the classical set. Furthermore, a NT "x" is denoted by <x, neut(x), anti(x)>. Also, many researchers have introduced NT structures in [12-20].

Matthew obtained partial metric spaces in [21]. The partial metric is generalization of classical metric space and it plays a significant role in fixed point theory and computer science. Also, many researchers studied partial metric space in [22-28].

Branciari obtained v-generalized metric in [29]. The v- generalized metric is a specific form of classical metric space for its triangular inequality. The most important use of

v-generalized metric space is fixed point theory. Furthermore, researchers studied v- generalized metric in [30- 34].

In this chapter, we obtain NT partial v-generalized metric space. In section 2; we give definitions of NT set in [11], NT partial metric space in [17] and NT v-generalized metric space in [18]. In section 3, we introduce NT partial v-generalized metric space and give some properties and examples for NT partial v-generalized metric space. Also, we show that NT partial v-generalized metric space is different from the NT partial metric space and NT v-generalized metric space. Furthermore, we show the relationship between NT partial metric spaces, NT v-generalized metric space with NT partial v-generalized metric space. Finally, we give definition of convergent sequence, Cauchy sequence and complete space for NT partial v-generalized metric space. In section 4, we give conclusions.

2. Basic and Fundamental Concepts

Definition 2.1: [29] Let N be a nonempty set and d:NxN$\rightarrow \mathbb{R}$ be a function. If d is satisfied the following properties, then it is called a v - generalized metric. For
n, m, $c_1, c_2, \ldots, c_v \in$ N,

i) d(n, m) ≥ 0 and d(n, m) $= 0 \Leftrightarrow$ n = m;

ii) d(n, m)= d(m, n);

iii) d(n, m) $\leq d$(n, c_1)+d(c_1, c_2)+d(c_2, c_3)+ $\ldots + d$(c_{v-1}, c_v)+ d(c_v, n). Where a, $c_1, c_2, \ldots, c_v$, b are all different.

Definition 2.2:[21]Let N be a nonempty set and d:NxN$\rightarrow \mathbb{R}$ be a function. If d is satisfied the following properties, then it is a partial metric. For p, r, s $\in$ N,

i) d(p, p) = d(s, s) = d(p, s) $\Leftrightarrow$ p = s;

ii) d(p, p) $\leq$ d(p, s);

iii) d(p, s) = d(s, p);

iv) d(p, r) $\leq$d(p, s)+d(s, r)-d(s, s);

Also, (N, d) is a partial metric space.

Definition 2.3: [11] Let # be a binary operation. (X, #) is a NT set (NTS) such

i) There must be neutral of "x" such x#neut(x) = neut(x)#x = x, x $\in$ X.

ii) There must be anti of "x" such x#anti(x) = anti(x)#x = neut(x), x $\in$ X.

Furthermore, a NT "x" is showed with (x, neut(x), anti(x)).

Also, neut(x) must different from classical unitary element.

Definition 2.4: [17] Let (M, *) be a NTS and m*n ∈ M for ∀ m, n ∈ M. NT partial metric (NTPM) is a function p: M x M → $\mathbb{R}^+ \cup \{0\}$ such for every s, p, r ∈ M,

i) p(m, n) ≥ p(n, n) ≥ 0

ii) If p_N(m, m) = p_N(m, n) = p_N(n, n) = 0, then there exits at least one m, n pair such that m = n.

iii) p(m, n) = p(n, m)

iv) If there exists at least an element n∈A for each m, k ∈ M pair such that

p(m, k) ≤ p(m, k*neut(n)), then p(m, k*neut(n)) ≤ p(m, n) + p(n, k) - p(n, n)

Also, ((N, *), p) is called NTPM space (NTPMS).

Definition 2.5: [13] A NT metric on a NTS (N, *) is a function d:NxN → $\mathbb{R}$ such that for every n, m, s ∈ N,

i) n * m ∈ N

ii) d(n, m) ≥ 0

iii) If n = m, then d(n, m) = 0

iv) d(n, m) = d(n, m)

v) If there exists at least an element s ∈ N for each n, m ∈ N pair such that

d(n, m) ≤ d(n, m*neut(s)), then d(n, m*neut(s)) ≤ d(n, s) + d_T(s, n).

Definiton 2.6: [18] Let (N, *) be a NTS. A NT v- generalized metric on N is a function d_v:NxN → $\mathbb{R}$ such that for every n, m, $k_1, k_2, \ldots, k_v$ ∈ N;

i) n*m ∈ N

ii) 0 ≤ d_v(n, m)

iii) if n = m, then d_v(n, m) = 0

iv) d_v(n, m) = d_v(m, n)

v) If there exists elements n, m, $k_1, \ldots, k_v$ ∈ N such that

d_v(n, m) ≤ d_v(n, m#neut(k_v)),

d_v(n, k_2)$\leq d_v$(n, k_2#neut(k_1)),

$d_v(k_1, k_3$)$\leq d_v(k_1, k_3$#neut(k_2)),

… ,

$d_v(k_{v-1}$, m)$\leq d_v(k_{v-1}$, m#neut(k_v));

then d_v(n, m*neut(k_v))$\leq$ d_v(n, k_1)+ $d_v(k_1, k_2)$ + … + $d_v(k_{v-1}, k_v)$ + $d_v(k_v$, m).

Where n, k_1, … , k_v, m are all different.

Furthermore, ((N, *), d_v) is called NTVGM space (NTVGMS).

3. Neutrosophic Triplet Partial v – Generalized Metric Space

Definition 3.1: Let (N, *) be a NTS. A NT partial v- generalized metric on N is a function d_{pv}:NxN$\rightarrow \mathbb{R}$ such every n, m, k_1, k_2, … , $k_v \in$ N;

i) n*m $\in$ N

ii) d_{pv}(n, m)$\geq d_{pv}$(n, n)≥ 0

iii) If d_{pv}(n, n) = d_{pv}(n, m) = d_{pv}(m, m) = 0, then n = m.

iv) d_{pv}(n, m) = d_{pv}(m, n)

v) If there exists elements n, m, k_1, … , $k_v \in$ N such that

d_{pv}(n, m)$\leq d_{pv}$(n, m*neut(k_v)),

d_{pv}(n, k_2)$\leq d_{pv}$(n, k_2*neut(k_1)),

$d_{pv}(k_1, k_3$)$\leq d_{pv}(k_1, k_3$*neut(k_2)),

… ,

$d_{pv}(k_{v-1}$, m)$\leq d_{pv}(k_{v-1}$, m*neut(k_v));

then

d_{pv}(n, m*neut(k_v))$\leq d_{pv}$(n, k_1)+ $d_{pv}(k_1, k_2)$ + … + $d_{pv}(k_{v-1}, k_v)$ + $d_{pv}(k_v$, m)-

[$d_{pv}(k_1, k_1)$ + $d_{pv}(k_2, k_2)$ + … + $d_{pv}(k_v, k_v)$].

Where n, k_1, … , k_v, m are all different.

Furthermore, ((N, *), d_{pv}) is called NTPVGM space (NTPVGMS).

Furthermore, if v = k, (k∈ N), then NTPVGMS is showed that NTPkGMS. For example, if v = 2, then NTPVGMS is showed that NTP2GMS.

Example 3.2: Let N = {∅, {k}, {l}, {k, l}} be a set and s(M) be number of elements in M ∈ N. Also, we can take neut(M)= M, anti(M)=M for all M ∈ N since M∪M = M, for M ∈ N. Furthermore, (N, ∪) is a NTS. Then, we take that d_{pv}: NxN→N is a function such that d_{pv}(S, K)= $2^{\max\{s(S),s(K)\}}$.

Now we show that d_{pv} is a NTPVGM.

i) S ∪ K ∈ N for S, K ∈ N.

ii) d_{pv}(S, K)= $2^{\max\{s(S),s(K)\}} \geq 2^{\max\{s(S),s(S)\}} \geq 0$

iii) There are not any elements S, K such that d_{pv}(S, K) = d_{pv}(S, S) = d_{pv}(K, K) = 0

 iv) d_{Pv}(S, K) = $2^{\max\{s(S),s(K)\}} = 2^{\max\{s(K),s(S)\}} = d_{pv}$(K, S)

v)

a) d_{pv}({l}, {k, l})≤ d_{pv}({l}, {k, l}∪ {k}) and d_{pv}({l, k}, ∅) ≤ d_{pv}({l, k}, ∅ ∪ {1}). Also,

d_{pv}({l}, {k, l}) = 2^2 = 4, d_{pv}({l}, {k}) = 2^1 = 2, d_{pv}({k}, ∅) = 2^1 = 2,

 d_{pv}(∅, {1, k}) = 2^2 = 4. Thus,

d_{pv}({l}, {k, l}∪{k})≤

d_{pv}({l}, {k})+d_{pv}({k}, ∅)+d_{pv}(∅, {1, k})- d_{pv}({k}, {k})-d_{pv}({1}, {1}).

b) d_{pv}({l}, {k})≤ d_{pv}({l}, {k}∪ {k, 1}) and d_{pv}({ k}, ∅) ≤ d_{pv}({k}, ∅ ∪ {1}). Also,

d_{pv}({l}, {k}) = 2^1 = 2, d_{pv}({l}, {k, 1}) = 2^2 = 4, d_{pv}({k, 1}, ∅) = 2^2 = 4,

d_{pv}(∅, {k})= 2^1 = 2. Thus,

d_{pv}({l}, {k}∪ {k,1})≤

d_{pv}({l}, {k,1})+d_{pv}({k,1}, ∅)+d_{pv}(∅, {k})- d_{pv}({k, 1}, {k,1})-d_{pv}({1}, {1}).

c) d_{pv}({l}, ∅)≤ d_{pv}({l}, ∅ ∪ {k, 1}) and d_{pv}(∅, { k}) ≤ d_{pv}(∅, { k}∪ {1}). Also,

$d_{pv}(\{l\}, \emptyset \cup \{k, 1\}) = 2^2 = 4, d_{pv}(\{l\}, \{k, 1\}) = 2^2 = 4, d_{pv}(\{k, 1\}, \{k\}) = 2^2 = 4,$
$d_{pv}(\{k\}, \{1\}) = 2^1 = 2.$ Thus,

$d_{pv}(\{l\}, \emptyset \cup \{k, 1\}) \leq$

$d_{pv}(\{l\}, \{k,1\}) + d_{pv}(\{k,1\}, \emptyset) + d_{pv}(\emptyset, \{k\}) - d_{pv}(\{k, 1\}, \{k,1\}) - d_{pv}(\{1\}, \{1\}).$

d) $d_{pv}(\{k\}, \emptyset) \leq d_{pv}(\{k\}, \emptyset \cup \{k, 1\})$ and $d_{pv}(\emptyset, \{l\}) \leq d_{pv}(\emptyset, \{1\} \cup \{k\}).$ Also,

$d_{pv}(\{k\}, \emptyset \cup \{k, 1\}) = 2^2 = 4, d_{pv}(\{k\}, \{k, 1\}) = 2^2 = 4, d_{pv}(\{k, 1\}, \emptyset) = 2^2 = 4,$

$d_{pv}(\emptyset, \{k\}) = 2^1 = 2.$ Thus,

$d_{pv}(\{k\}, \emptyset \cup \{k, 1\}) \leq$

$d_{pv}(\{k\}, \{k, 1\}) + d_{pv}(\{k,1\}, \emptyset) + d_{pv}(\emptyset, \{k\}) - d_{pv}(\{k, 1\}, \{k,1\}) - d_{pv}(\{k\}, \{k\}).$

e) $d_{pv}(\{1, k\}, \emptyset) \leq d_{pv}(\{1, k\}, \emptyset \cup \{k\})$ and $d_{pv}(\emptyset, \{l, k\}) \leq d_{pv}(\emptyset, \{l, k\} \cup \{l\}).$ Also,

$d_{pv}(\{1, k\}, \emptyset \cup \{k\}) = 2^2 = 4, d_{pv}(\{1, k\}, \{k\}) = 2^2 = 4, d_{pv}(\{k\}, \{1\}) = 2^1 = 2,$

$d_{pv}(\{1\}, \emptyset) = 2^1 = 2.$ Thus,

$d_{pv}(\{1, k\}, \emptyset \cup \{k\}) \leq$

$d_{pv}(\{1, k\}, \{k\}) + d_{pv}(\{k\}, \{1\}) + d_{pv}(\{1\}, \emptyset) - d_{pv}(\{k\}, \{k\}) - d_{pv}(\{1\}, \{1\}).$

f) $d_{pv}(\{k\}, \{1, k\}) \leq d_{pv}(\{k\}, \{1, k\} \cup \{1\})$ and $d_{pv}(\{l, k\}, \emptyset) \leq d_{pv}(\{l, k\}, \emptyset \cup \{k\}).$ Also,

$d_{pv}(\{k\}, \{1, k\} \cup \{1\}) = 2^2 = 4, d_{pv}(\{k\}, \{1\}) = 2^1 = 2, d_{pv}(\{1\}, \emptyset) = 2^1 = 2,$

$d_{pv}(\{\emptyset, \{1,k\}) = 2^2 = 4.$ Thus,

$d_{pv}(\{k\}, \{1, k\} \cup \{1\}) \leq$

$d_{pv}(\{k\}, \{1\}) + d_{pv}(\{1\}, \emptyset) + d_{pv}(\{\emptyset, \{1,k\}) - d_{pv}(\{1\}, \{1\}) - d_{pv}(\{k\}, \{k\}).$

Hence, (X), $\cup$), d_{pv}) is a NT2GMS.

Corollary 3.3: NTPVGMS is different from the partial metric space and NTPMS, since for triangle inequality and * binary operation.

Corollary 3.4: The NTPVGMS is different from NTVMS since for triangle inequality and condition iii in Definition 3.1.

Theorem 3.5: In Definition 3.1, if it is taken such that $k_1 = k_2 = \ldots = k_v$, and $d_{pv}(n, n) = 0$, then each NTPVGMS is a NTVGMS.

Proof: Let $((X, \#), d_{pv})$ be a NTPVGMS.

i) a#b $\in$ X since $((X, \#), d_{pv})$ is a NTPVGMS.

ii) $0 \leq d_{pv}(n, m)$ since $((X, \#), d_{pv})$ is a NTPVGMS.

iii) If $n = m$, then $d_{pv}(n, m) = 0$; because $d_{pv}(n, n) = 0$

iv) $d_v(n, m) = d_v(m, n)$ since $((X, \#), d_{pv})$ is a NTPVGMS.

v) From Definition 3.1,

If there exists elements n, m, $k_1, \ldots, k_v \in$ N such that

$d_{pv}(n, m) \leq d_{pv}(n, m\#neut(k_v))$,

$d_{pv}(n, k_2) \leq d_{pv}(n, k_2\#neut(k_1))$,

$d_{pv}(k_1, k_3) \leq d_{pv}(k_1, k_3\#neut(k_2))$,

$\ldots$,

$d_{pv}(k_{v-1}, m) \leq d_{pv}(k_{v-1}, m\#neut(k_v))$;

Then,

$d_{pv}(n, m*neut(k_v)) \leq$

$d_{pv}(n, k_1) + d_{pv}(k_1, k_2) + \ldots + d_{pv}(k_{v-1}, k_v) + d_{pv}(k_v, m) -$

$[d_{pv}(k_1, k_1) + d_{pv}(k_2, k_2) + \ldots + d_{pv}(k_v, k_v)]$ (1)

Then we take $k_1 = k_2 = \ldots = k_v$, and $d_{pv}(n, n) = 0$ in (1). Thus, for a, b, $u_1 \in$ X,

$d_{pv}(n, m) \leq d_{pv}(n, m\#neut(k_1))$

$d_{pv}(n, k_1) \leq d_{pv}(n, k_1\#neut(k_1))$,

$d_{pv}(k_1, k_1) \leq d_{pv}(k_1, k_1\#neut(k_1))$,

$\ldots$,

$d_{pv}(k_1, \text{m}) \leq d_{pv}(k_1, \text{m\#neut}(k_1));$

then

$d_{pv}(\text{n, m*neut}(k_v)) = d_{pv}(\text{n, m*neut}(k_1)) \leq$

$d_{pv}(\text{n}, k_1) + d_{pv}(k_1, k_1) + \dots + d_{pv}(k_1, k_1) + d_{pv}(k_1,\text{m}) -$

$[d_{pv}(k_1, k_1) + d_{pv}(k_1, k_1) + \dots + d_{pv}(k_1, k_1)] = d_{pv}(\text{n}, k_1) + 0 + d_{pv}(k_1,\text{m}) - 0 =$

$d_{pv}(\text{n}, k_1) + d_{pv}(k_1,\text{n}).$

Therefore, $(X, \#), d_v)$ is a NTVGMS.

Corollary 3.6: In Theorem 3.5, we can define a NTP1GMS with each NTVGMS. Also, from Theorem 3.5, each NTVGMS is a NTP1GMS.

Theorem 3.7: Let $((X, \#), d)$ be a NTVGMS and d_v be a function such that

$d_{pv}(\text{n, m}) = d(\text{n, m}) + \text{k} \ (\text{k} \in \mathbb{R}^+)$. Then $((X, \#), d_{pv})$ is a NTPVGMS.

Proof:

We take $d_{pv}(\text{n, m}) = d(\text{n, m}) + \text{k}$. Where, $d_{pv}(\text{n, n}) = d(\text{n, n}) + \text{k} = \text{k}$ since $((X, \#), d)$ is a NTVGMS.

 i) It clear that a#b $\in$ X since $((X, \#), d)$ is a NTVGMS.

ii) $d_{pv}(\text{n, m}) = d(\text{n, m}) + \text{k} \geq d_{pv}(\text{n, n}) = \text{k} \geq 0$ since $((X, \#), d)$ is a NTVGMS.

iii) There is not any pair of element n, m such that

$d_{pv}(\text{n, m}) = d(\text{n, m}) + \text{k} = d_{pv}(\text{n, n}) = \text{k} = d_{pv}(\text{m, m}) = \text{k} = 0$. Because, $\text{k} \in \mathbb{R}^+$.

iv) $d_{pv}(\text{n, m}) = d(\text{n, m}) + \text{k} = d(\text{m, n}) + \text{k} = d_{pv}(\text{m, n})$

v) In Definition 2.6, If there exists elements n, m, $k_1, \dots, k_v \in$ X such that

 $d(\text{n, m}) \leq d(\text{n, m\#neut}(k_v)),$

 $d(\text{n}, k_2) \leq d(\text{n}, k_2\text{\#neut}(k_1)),$

 $d(k_1, k_3) \leq d(k_1, k_3\text{\#neut}(k_2)),$

 $\dots,$

 $d(k_{v-1}, \text{m}) \leq d(k_{v-1}, \text{m\#neut}(k_v));$

then

d(n, m*neut(k_v))≤

d(n, k_1)+d(k_1, k_2)+ … +d(k_{v-1}, k_v)+d(k_v, m)   (2)

As d_{pv}(n, m) = d(n, m) + k, we can take in (2),

d_{pv}(n, m) = d(n, m) + k ≤ d_{pv}(n, m#neut(k_v)) = d(n, m#neut(k_v)) + k,

d_{pv}(n, k_2) = d(n, k_2) + k ≤ d_{pv}(n, k_2#neut(k_1)) = d(n, k_2#neut(k_1)) + k

d_{pv}(k_1, k_3) = d(k_1, k_3) + k ≤ d_{pv}(k_1, k_3#neut(k_2)) = d(k_1, k_3#neut(k_2)) + k,

… ,

d_{pv}(k_{v-1}, m) = d(k_{v-1}, m) + k ≤ d_{pv}(k_{v-1}, m#neut(k_v)) = d(k_{v-1}, m#neut(k_v)) + k;

then

d_{pv}(n, m*neut(k_v)) = d(n, m*neut(k_v)) + k ≤   d(n, k_1)+ k + d(k_1, u_2) +k + … + d(k_{v-1}, k_v) +k + d(k_v, m) +k. Thus,

d_{pv}(n, m*neut(k_v)) = d(n, m*neut(k_v)) + k ≤

d(n, k_1)+ d(k_1, k_2) + … + d(k_{v-1}, k_v) + d(k_v, m)-v.k. Therefore,

d_{pv}(n, m*neut(k_v))≤ d_{pv}(n, k_1)+ d_{pv}(k_1, k_2) + … + d_{pv}(k_{v-1}, k_v) + d_{pv}(k_v, m)-

[d_{pv}(k_1, k_1) + d_{pv}(k_2, k_2) + … + d_{pv}(k_v, k_v)] since d_v(n, n) = k for all n ∈ X.

Corollary 3.8: In Theorem 3.7, we can define a NTPVGMS with each NTVGMS.

Definition 3.9: Let ((X, #), d_v) be a NTPVGMS and {x_n} be a sequence in NTPVGMS and m ∈ X. If there exist N ∈ ℕ for every ε>0 such that

d_v(m, {x_n}) < ε + d_v(m, m),

then {x_n} converges to m. where n ≥M. Also, it is showed that

$\lim_{n\to\infty} x_n$= m or x_n→ m.

Definition 3.10: Let ((X, #), d_v) be a NTPVGMS and {x_n} be a sequence in NTPVGMS. If there exist a N ∈ ℕ for every ε>0 such that

$d_v(\{x_m\}, \{x_n\}) < \varepsilon + d_v(m, m),$

then $\{x_n\}$ is a Cauchy sequence in NTPVGMS. Where, n $\geq$ m$\geq$M and m $\in$ X.

Definition 3.11: Let $((X, \#), d_v)$ be a NTPVGMS and $\{x_n\}$ be a sequence in NTPVGMS. If there exist N $\in \mathbb{N}$ for every $\varepsilon > 0$ such that

$d_v(\{x_n\}, \{x_{n+1+jk}\}) < \varepsilon + d_v(m, m), \quad$ (j = 0, 1, 2, …)

then $\{ x_n \}$ is a k - Cauchy sequence in NTPVGMS. Where, k $\in \mathbb{N}$ and m $\in$ X.

Definition 3.12: Let $((X, \#), d_v)$ be a NTPVGMS and $\{x_n\}$ be Cauchy sequence in NTPVGMS. NTPVGMS is complete $\Leftrightarrow$ every $\{x_n\}$ converges in NTPVGMS.

Definition 3.13: Let $((X, \#), d_v)$ be a NTPVGMS and $\{x_n\}$ be k - Cauchy sequence in NTPVGMS. NTPVGMS is k - complete $\Leftrightarrow$ every $\{x_n\}$ converges in NTPVGMS.

Conclusions

In this chapter, we obtained NTPVGMS. We also show that NTPVGMS is different from the NTVGMS and NTPMS. Also, we defined complete space and k-complete space for NTPIPS. Thus, we have added a new structure to NT structure and we gave rise to a new field or research called NTPIPS. Also, thanks to NTPIPS researcher can obtain new structure and properties. For example, NT partial v – generalized normed space, NT partial v – generalized inner product space and fixed point theorems for NTVGMS.

Abbreviations

NT: Neutrosophic triplet

NTS: Neutrosophic triplet set

NTM: Neutrosophic triplet metric

NTMS: Neutrosophic triplet metric space

NTPM: Neutrosophic triplet partial metric

NTPMS: Neutrosophic triplet partial metric space

NTVGM: Neutrosophic triplet v-generalized metric

NTVGMS: Neutrosophic triplet v-generalized metric space

NTPVGM: Neutrosophic triplet partial v-generalized metric

NTPVGMS: Neutrosophic triplet partial v-generalized metric space

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Chapter Three

NEUTROSOPHIC TRIPLET R-MODULE

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Abstract

In this chapter, neutrosophic triplet(NT) R-module is presented and the properties of NT R-module are studied. Additionaly, we conclude that the NT R-module is different from the classical R-module. Then, we compared NT R-module with the NT vector space and NT G – module as well.

Keywords: NT vector space, NT G – module, NT R – module

1. Introduction

In 1980, Smarandache presented neutrosophy which is a part of philosophy. Neutrosophy depends on neutrosophic logic, probability and set in [1]. Neutrosophic logic is a general concept of some logics such as fuzzy logic that is presented by Zadeh in [2] and intuitionistic fuzzy logic that is presented by Atanassov in [3]. Fuzzy set has the function of membership but intuitionistic fuzzy set has the function of membership and function of non-membership and they don't describe the indeterminancy states. However; neutrosophic set includes these all functions. A lot of researchers have studied the concept of neutrosophic theory and its application to multi-criteria decision making problems in [4-11]. Sahin M., and Kargın A., investigated NT metric space and NT normed space in [12]. Lately, Olgun at al. introduced the neutrosophic module in [13]; Şahin at al. presented Neutrosophic soft lattices in [14]; soft normed rings in [15]; centroid single valued neutrosophic triangular number and its applications in [16]; centroid single valued neutrosophic number and its applications in [17]. Ji at al. searched multi – valued neutrosophic environments and its applications in [18]. Also, Smarandache at al. searched NT theory in [19] and NT groups in [20, 21]. A NT has a form <m, neut(m), anti(m)> where; neut(m) is neutral of "m" and anti(m) is opposite of "m". Moreover, neut(m) is different from the classical unitary element and NT group is different from the classical group as well. Lately, Smarandache at al. investigated the NT field [22] and the NT ring [23]. Şahin at al. presented NT metric space, NT vector space and NT normed space in [24] and NT inner product in [25]. Smarandache at al. searched NT G- Module in [26]. Bal

at al. searched NT cosets and quotient groups in [27]. Şahin at al. presented fixed point theorem for NT partial metric space and Neutrosophic triplet v – generalized metric space in [28-29] and Çelik at al. searched fundamental homomorphism theorems for NETGs in [30].

The concept of an R – module over a ring is a stereotype of the notion of vector space, where the corresponding scalars are allowed to lie in an arbitrary ring. As the basic structure of the abelian ring can significantly be more complex and displeasing than the structure of a field, the theory of modules are much more complex than the structure of a vector space. Lately, Ai at al. defined the irreducible modules and fusion rules for parafermion vertex operator algebras in [31] and Creutzig at al. introduced Braided tensor categories of admissible modules for affine lie algebras in [32].

In this work, we study the concept of NT R-Modules. So we obtain a new algebraic structures on NT groups and NT ring. In section 2, we give basic definitions of NT sets, NT groups, NT ring, NT vector space and NT G-modules. In section 3, we define NT R-module and we present some properties of a NT R-module. We point out that NT R-module is different from the classical R-module. Also, we define NT R-module homomorphism and NT coset for NT R – module. Additionally, we describe NT quotient R – module. Finally, we give some results in section 4.

2. Preliminaries

Definition 2.1: [21] Let N be a set together with a binary operation ∇. Then, N is called a NT set if for any $k \in N$ there exists a neutral of "k" called $neut(k)$ that is different from the classical algebraic unitary element and an opposite of "k" called $anti(k)$ with $neut(k)$ and $anti(k)$ belonging to N, such that

$$k \nabla neut(k) = neut(k) \nabla k = k,$$

and

$$k \nabla anti(k) = anti(k) \nabla k = neut(k).$$

Definition 2.2: [21] Let (N, ∇) be a NT set. Then, N is called a NT group if the following conditions hold.

(1) If (N, ∇) is well-defined, i.e., for any $k, l \in N$, one has $k \nabla l \in N$.

(2) If (N, ∇) is associative, i.e., $(k \nabla l)\nabla m = k \nabla (l \nabla m)$ for all $k, l, m \in N$.

Definition 2.3: [24] Let $(NTF, \nabla_1, \blacksquare_1)$ be a NT field, and let $(NTV, \nabla_2, \blacksquare_2)$ be a NT set together with binary operations "∇_2" and "$\blacksquare_2$". Then $(NTV, \nabla_2, \blacksquare_2)$ is called a NT vector space if the following conditions hold. For all $p, r \in NTV$, and for all $t \in NTF$, such that $p\nabla_2 r \in NTV$ and $p\blacksquare_2 t \in NTV$ [24];

(1) $(p\nabla_2 r)\nabla_2 s = p\nabla_2 (r\nabla_2 s); p, r, s \in NTV$;

(2) $p\nabla_2 r = r\nabla_2 p; p, r \in NTV$;

(3) $(r\nabla_2 p)\blacksquare_2 t = (r\blacksquare_2 t)\nabla_2 (p\blacksquare_2 t); t \in NTF \text{ and } p, r \in NTV$;

(4) $(t\nabla_1 c)\blacksquare_2 p = (t\blacksquare_2 p)\nabla_1 (c\blacksquare_2 p); t, c \in NTF \text{ and } p \in NTV$;

(5) $(t\blacksquare_1 c)\blacksquare_2 p = t\blacksquare_1 (c\blacksquare_2 p); t, c \in NTF \text{ and } p \in NTV$;

(6) There exists any $t \in NTF \ni p\blacksquare_2 neut(t) = neut(t)\blacksquare_2 p = p; p \in NTV$.

Definition 2.4: [26] Let (G, ∇) be a NT group, $(NTV, \nabla_1, \blacksquare_1)$ be a NT vector space on a NT field $(NTF, \nabla_2, \blacksquare_2)$, and $g \nabla l \in NTV$ for $g \in G, l \in NTV$. If the following conditions are satisfied, then $(NTV, \nabla_1, \blacksquare_1)$ is called NT G-module.

(a) Thereexists $g \in G \ni k * neut(g) = neut(g) * k = k$, for every $k \in NTV$;

(b) $l\nabla_1 (g\nabla_1 h) = (l\nabla_1 g)\nabla_1 h, \quad \forall l \in NTV; g, h \in G$;

(c) $(r_1\blacksquare_1 s_1\nabla_1 r_2\blacksquare_1 s_2)\nabla g = x\blacksquare_1 (h\nabla g)\nabla_1 y\blacksquare_1 (l\nabla g), \forall x, y \in NTF; h, l \in NTV; g \in G$.

Definition 2.5: [23] The NT ring is a set endowed with two binary laws $(M, *, \#)$ such that,

a) $(M, *)$ is a abelian NT group; which means that:

- $(M, *)$ is a commutative NT with respect to the law * (i.e. if x belongs to M, then $neut(x)$ and $anti(x)$, defined with respect to the law *, also belong to M)

- The law * is well – defined, associative, and commutative on M (as in the classical sense);

b) $(M, *)$ is a set such that the law # on M is well-defined and associative (as in the classical sense);

c) The law is distributive with respect to the law * (as in the classical sense)

3. Neutrosophic Triplet R-Module

In this section, we define the NT R-module, NT R-submodule, NT cosets, NT quotient R-module, and NT R-module homomorphism. Then, we point out that NT R-module has more properties than the classical R – module.

Definition 3.1: Let (NTR,∇, $\blacksquare$) be a commutative NT ring and let (NTM, $*$) be a NT abelian group and $\circ$ be a binary operation such that $\circ$: NTR x NTM$\rightarrow$ NTM. Then (NTM, $*$, $\circ$) is called a NT R-Module on (NTR,∇,$\blacksquare$) if the following conditions are satisfied. where,

1) p $\circ$(r$*$s) =(p $\circ$r)$*$ (p $\circ$s), $\forall$ r, s $\in$ NTM and p $\in$ NTR.

2) (p∇k)$\circ$r = (p∇r)$\circ$(k∇r), $\forall$ p, k $\in$ NTR and $\forall$ r $\in$ NTM

3) (p$\blacksquare$k)$\circ$r = p$\blacksquare$(k$\circ$r), $\forall$ r, s $\in$ NTR and $\forall$ m $\in$ NTM

4) For all m $\in$ NTM; there exists at least a c $\in$ NTR such that m$\circ$neut(c)= neut(c)$\circ$m = m. Where, neut(c) is neutral element of c for $\blacksquare$.

Example 3.2: Let A={x, y} be a set and P(A)={ $\emptyset$, {x}, {y}, {x, y}} be power set of A. Hence, from Definition 2.2, (P(A), $\cup$) is a NT commutative group such that for neut(B)= B, anti(B)= B. Also, from Definition 2.5, (P(A), $\cup$, $\cap$) is a NT ring since for neut(B)= B, anti(B)= B for "$\cup$, $\cap$". Now, we show that (P(A), $\cup$, $\cap$) is a NT ring on (P(B), $\cup$, $\cap$). Where, it is clear that $\cup$ is a binary operation such that $\cap$: P(B) x P(B)$\rightarrow$ P(B)

1) It is clear that K$\cap$(L$\cup$M) = (K$\cap$L)$\cup$(K$\cap$M), for all ;K, L, M $\in$ P(A)

2) It is clear that (K$\cup$L)$\cap$M= (K$\cap$M)$\cup$(L$\cap$M), for all K, L, M $\in$ P(A).

3) (K$\cap$L)$\cap$M= K$\cap$(L$\cap$M) for all K, L, M $\in$ P(A) .

4) For all B$\in$ P(A); such that B $\cap$neut(B)=neut(B) $\cap$B=B, there exist neut(B)= B $\in$ P(A).

 Therefore, (P(A), $\cup$, $\cap$) is a neutrosophic triplet R-module on (P(A), $\cup$, $\cap$).

Corollary 3.3: In condition 4) of Definition 3.1, neut(c) need not be unique. Thus, NT R-Module is generally different from the classical R-Module.

Corollary 3.4: From Definition 3.1 and Definition 2.3, a NT vector space is a NT R-module, But a NT R-module is not generally a NT vector space.

Note 3.5: Let (NTR,∇, $\blacksquare$) be a commutative NT ring. In this paper, we define $neut_\nabla$(a) is neutral element of a for binary operation ∇,

$anti_\nabla$(a) is anti element of a for binary operation ∇. Also,

$neut_\blacksquare$(a) is neutral element of a for binary operation $\blacksquare$,

$anti_\blacksquare$(a) is anti element of a for binary operation $\blacksquare$.

Proposition 3.6: Let (NTM, $*$, °) be a NT R-Module on NT ring (NTR,∇,■). Then, for all m ∈ NTM and c ∈ NTR, there exists at least a n ∈ NTM such that

$$neut_\nabla(c)°m = neut_\nabla(n),$$

Proof: From properties of NT gorup, it is clear that

$$neut_\nabla(c)°m = (neut_\nabla(c)\nabla\ neut_\nabla(c))\ °m \tag{1}$$

Also, from Definition 3.1,

$$neut_\nabla(c)°m = (neut_\nabla(c)\nabla\ neut_\nabla(c))\ °m = (neut_\nabla(c)\ °m)\ \nabla\ (neut_\nabla(c)\ °m) \tag{2}$$

Furthermore, from Definition 3.1, it is clear that

$$neut_\nabla(c)°m \in NTM \tag{3}$$

From 3), we take $neut_\nabla(c)°m = n$. Thus, from (2), there exists at least a n∈ NTM such that n = n∇n. Therefore, n = $neut_\nabla(n)$. Then we obtain $neut_\nabla(c)°m = neut_\nabla(n)$.

Definition 3.7: Let (NTM, $*$, °) be a NT R-Module on NT ring (NTR,∇,■) and NTSM ⊂ NTM. Then (NTSM, $*$, °) is called NT R - submodule of (NTM, $*$, °), if (NTSM, $*$, °) is a NT R – module on NT ring (NTR,∇,■).

Example 3.8: In example 3.2, for P(A)={ ∅, {x}, {y}, {x, y}}, (P(A), ∪, ∩) is a NT R-module on NT ring (P(A), ∪, ∩). Also, S = {∅, {x}} ⊂ P(A) and it is clear that   (S, ∪, ∩) is a NT R-module on (P(A), ∪, ∩). Thus, (S, ∪, ∩) is a NT R-submodule of (P(A), ∪, ∩).

Theorem 3.9: Let (NTM, $*$, °) be a NT R-Module on NT ring (NTR,∇,■) and NTSM be a NT subgroup of NTM. Then, (NTSM, $*$, °) is a NT R-submodule of (NTM, $*$, °) if and only if the following conditions hold.

i) NTSM ≠ ∅

ii)For x, y ∈ NTSM, m, n ∈ NTR; (x°m) $*$ (y°n) ∈ NTSM

Proof: (⇒) If NTSM is a NT R-submodule of (NTM, $*$, °), from Definition 3.1 and definition 3.6, i) and ii) are hold.

(⇐)If condition ii) is hold, then we can take °: NTR x NTSM→ NTSM. Also, NTS is satisfied the condition of Definition 3.1 since NTSM is a NT subgroup of NTM and from i). Thus, (NTSM, $*$, °) is a NT R-submodule of (NTM, $*$, °).

Theorem 3.10: Let (NTM, $*$, °) be a NT R-Module on NT ring (NTR,∇,■). Then, (NTM, $*$, °) is a NT R-module on (NTM, $*$, °), if and only if the following conditions are satisfied.

1) m°s ∈ NTM

2) c °(m$*$s) =(c °x)$*$ (c °s), for all m, s, c∈ NTM

3) (c$*$t)°m = (c°m)$*$(t°m), for all c, t, m ∈ NTM

4) (c°t)°m = c°(t°m) for all for all c, t, m ∈ NTM

5) ∀ m ∈ NTM; there exists at least a s ∈ NTR such that m°neut(s)= neut(s)°m = m. Where, neut(s) is neutral element of s for °.

Proof: (⇒)We assume that (NTM, ∗, °) is a NT R-module on (NTM, ∗, °). Thus, (NTM, ∗, °) is NT ring and we can we take ∇ = ∗, ∎ = ° and   NTM = NTR. Also, from Definition 3.1, it is clear that (NTM, ∗, °) satisfies the conditions 1, 2, 3, 4 and 5.

(⇐) We assume that conditions 1, 2, 3, 4 and 5 are hold. As (NTM, ∗, °) is a NT R-module, it is clear that (NTM, ∗) is a abelian NT group. As conditions 1 is hold, we can take °: NTM→ NTM. Also, (NTM, ∗, °) is a NT ring from   definition 2. 5 since conditions 1, 2, 3, 4 are hold. Then, if we take ∇ = ∗, ∎ = °, from Definition 3.1, (NTM, ∗, °) is a NT R-module on (NTM, ∗, °).

Corollary 3.11: Let (NTM, ∗, °) be a NT R-Module on NT ring (NTR,∇,∎). Then, (NTM, ∗, °) is a NT R-module on (NTM, ∗, °), if and only if the following conditions are satisfied.

1) (NTM, ∗, °) is a NT ring.

2) For all m ∈ NTM; there exists at least a c ∈ NTR such that m°neut(c) = neut(c)°m = m. Where, neut(c) is neutral element of c for °.

Proof: (NTM, ∗) is a abelian NT group since (NTM, ∗, °) be a NT R-Module. So, in Theorem 3.8, conditions 1, 2, 3, 4 are conditions of NT ring. Also, condition 2) is equal conditions 5) in Theorem 3.9. Thus, from Theorem 3.9 the proof is clear.

Definition 3.12: Let (NTM, ∗, °) be a NT R-module on NT ring (NTR,∇,∎) and Let (NTSM, ∗, °) be a NT R-submodule of (NTM, ∗, °). The NT cosets of (NTSM, ∗, °) in (NTM, ∗, °) are denoted by

m*NTSM, for all m ∈ NTM.

Also, for m*NTSM and n*NTSM cosets of (NTSM, ∗, °) in (NTM, ∗, °),

(m*NTSM) * (n*NTSM) = (m*n)*NTSM

Definition 3.13: Let (NTM, ∗, °) be a NT R-module on NT ring (NTR,∇,∎) and Let (NTSM, ∗, °) be a NT R-submodule of (NTM, ∗, °). The NT quotient R-module of (NTM, ∗, °) is denoted by (NTM, ∗, °)/(NTSM, ∗, °) such that

(NTM, ∗, °)/(NTSM, ∗, °) = {m*NTSM: m ∈ NTM}

Where, it is clear that NTM/NTSM is a set of NT cosets of (NTSM, ∗, °) in (NTM, ∗, °).

Example 3.14: In Example 3.2, for P(A) = { ∅, {x}, {y}, {x, y}}, (P(A), ∪, ∩) is a NT R-module on NT ring (P(A), ∪, ∩). Also, S = {∅, {x}} ⊂ P(A) and (S, ∪, ∩) is a NT R-submodule of (P(A), ∪, ∩). Now, we give NT cosets of (S,∪, ∩) in (P(A), ∪, ∩).

$\emptyset \cup S = \{\{\emptyset, \{x\}\}$

$\{x\} \cup S = \{\{x\}\}$

$\{y\} \cup S = \{\{y\}, \{x, y\}\}$

$\{x, y\} \cup S = \{\{x, y\}\}.$

Also, $(P(A), \cup, \cap)/(S, \cup, \cap) = \{\{\{\emptyset, \{x\}\}, \{\{x\}\}, \{\{y\}, \{x, y\}\}, \{\{x, y\}\}\}.$

Definition 3.15: $(NTM_1, *_1, \circ_1)$ be a NT R-module on NT ring $(NTR, \nabla, \blacksquare)$ and $(NTM_2, *_2, \circ_2)$ be a NT R-module on NT ring $(NTR, \nabla, \blacksquare)$. A mapping f: $NTM_1 \rightarrow NTM_2$ is said to be NT R-module homomorphism when

$f((r\circ_1 m) *_1 (s\circ_1 n)) = (r\circ_2 f(m)) *_2 (s\circ_2 f(n))$, for all r, s $\in$ NTR and m, n $\in NTM_1$.

Conclusion

In this work; we presented NT R-module. We defined NT R-module by using the NT group and NT ring. Moreover, we show that NT R – module is different from the classical R – module. We show that NT R - module has new properties compared to the classical G - module. Finally, by using NT R - module, theory of representation of NT rings can be defined and the applications of NT structures will be expanded.

Abbreviations

NT: Neutrosophic triplet

NTS:Neutrosophic triplet set

NETG: Neutrosophic extended triplet group

NTM: Neutrosophic triplet R-module

NTSM: Neutrosophic triplet R-submodule

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Chapter Four

Neutrosophic Triplet Topology

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Abstract

The neutrosophic triplet structures are new concept in neutrosophy. Furthermore, topology has many different application areas in mathematic. In this chapter, we obtain neutrosophic triplet topology. Then, we give some definitions and examples for neutrosophic triplet topology and we obtain some properties and prove these properties. Finally, we show that neutrosophic triplet topology is different from classical topology.

Keywords: Neutrosophic triplet set, neutrosophic triplet structures, neutrosophic triplet topology

1. Introduction

Florentin Smarandache introduced neutrosophy in 1980, which studies a lot of scientific fields. In neutrosophy, there are neutrosophic logic, set and probability in [1]. Neutrosophic logic is a generalization of a lot of logics such as fuzzy logic in [2] and intuitionistic fuzzy logic in [3]. Neutrosophic set denoted by (t, i, f) such that "t" is degree of membership, "i" is degree of indeterminacy and "f" is degree of non-membership. Many researchers have studied neutrosophic sets in [4-9,23-27]. Furthermore, Florentin Smarandache and Mumtaz Ali obtained neutrosophic triplet (NT) in [10] and they introduced NT groups in [11]. For every element "x" in neutrosophic triplet set A, there exist a neutral of "a" and an opposite of "a". Also, neutral of "x" must different from the classical neutral element. Therefore, the NT set is different from the classical set. Furthermore, a NT "x" is denoted by <x, neut(x), anti(x)>. Also, many researchers have introduced NT structures in [11-19].

Topology is a branch of mathematic that deals with the specific definitions given for spatial structure concepts, compares different definitions and explores the connections between the structures described on the sets. In mathematics it is a large area of study with many more specific subfields. Subfields of topology include algebraic topology, geometric topology, differential topology, and manifold topology. Thus, topology has many different application areas in mathematic. For example, a curve, a surface, a family of curves, a set

of functions or a metric space can be a topology space. Also, the topology has been studied on neutrosophic set, fuzzy set, intuitionistic fuzzy set and soft set. Many researchers have introduced the topology in [20-22].

In this chapter, we introduce NT topology. In section 2, we give definition of NT set in [11]. In section 3, we introduce NT topology and we give some properties and examples for NT topology. Also, we define open set, close set, inner point, a set of inner point, outside point, a set of outside, closure point and a set of closure in a NT topology. In section 4, we give some conclusions.

2. Basic and Fundamental Concepts

Definition 2.1: [11]

Let # be a binary operation. (X, #) is a NT set (NTS) such

i) There must be neutral of "x" such x#neut(x) = neut(x)#x = x, x ∈ X.

ii) There must be anti of "x" such x#anti(x) = anti(x)#x = neut(x), x ∈ X.

Furthermore, a NT "x" is showed with (x, neut(x), anti(x)).

Also, neut(x) must different from classical unitary element.

3. Neutrosophic Triplet Topology

Definition 3.1: Let (X, *) be a NT set, P(X) be set family of each subset of X and $\mathcal{T}$ be a subset family of P(X). If $\mathcal{T}$ is satisfied the following conditions, then $\mathcal{T}$ is called a NT topology on X.

i) A*B ∈ X, A, B ∈ X

ii) ∅, X ∈ $\mathcal{T}$

iii) For ∀ i ∈ K, If A_i ∈ X, then $\bigcup_{i \in K} A_i$ ∈ $\mathcal{T}$

iv) For ∀ i ∈ K (K is finite), If A_i ∈ X, then $\bigcap_{i \in K} A_i$ ∈ $\mathcal{T}$

   Also, ((X, *), $\mathcal{T}$) is called NT topology space.

Example 3.2: Let X = {k, l} be set and P(X) = {∅, {k}, {l}, {k, l}}. We can take A∪A= A for A ∈ X. Thus, we can take

neut(A)= A, anti(A)=A. Then, (P(X), ∪) is a NT set. Also,

i) A∪B ∈ P(X) for A, B ∈ P(X),

ii) Ø, X ∈ 𝒯

iii) For ∀ i ∈ K, If A_i ∈ X, then $\bigcup_{i \in K} A_i$ ∈ 𝒯.

iv) For ∀ i ∈ K (K is finite), If A_i ∈ X, then $\bigcap_{i \in K} A_i$ ∈ 𝒯.

Thus, 𝒯 = P(X) is a NT topology on X and ((X, ∪), 𝒯) is called NT topology space.

Furthermore, If X is an arbitrary set and P(X) is set family of each subset of X, then (P(X), ∪) is a NT set and 𝒯 = P(X) is a NT topology and ((X, ∪), 𝒯) is called NT topology space.

Example 3.3: Let X = {x, y} be a set and 𝒯 = P(X) = {Ø, {x}, {y}, {x, y}}. We can take A∩A= A, for A ∈ X. Thus, we can take

 neut(A)= A, anti(A)=A. Then, (P(X), ∩) is a NT set. Also,

i) A∩B ∈ P(X), for A, B ∈ 𝒯,

ii) Ø, X ∈ 𝒯

iii) For ∀ i ∈ K, If A_i ∈ X, then $\bigcup_{i \in K} A_i$ ∈ 𝒯.

iv) For ∀ i ∈ K (K is finite), If A_i ∈ X, then $\bigcap_{i \in K} A_i$ ∈ P(X).

Thus, 𝒯 = P(X) is a NT topology and ((X, ∩), 𝒯) is called NT topology space.

Furthermore, If X is an arbitrary set and 𝒯= P(X) is set family of each subset of X, then (P(X), ∪) is a NT set and 𝒯 = P(X) is a NT topology. Thus, ((X, ∩), 𝒯) is called NT topology space.

Theorem 3.4: Let $(𝒯_i)$ (i ∈ K) be a family of NT topologies on nonempty NT set (X, #). ∩ $𝒯_i$ is a NT topology.

Proof:

i) Since $(𝒯_i)$ (i ∈ K) is a family of NT topologies on nonempty NT set (X, #), it is clear that for A, B ∈ X , A#B ∈ X.

ii) Since $(𝒯_i)$ (i ∈ K) is a family of NT topologies on nonempty NT set (X, #)

 X, Ø ∈ $𝒯_i$. Thus, it is clear that X, Ø ∈ ∩ $𝒯_i$.

iii) Let A_i ∈ ∩ $𝒯_i$ (i ∈ I). It is clear that A_i ∈ $𝒯_i$. Since $(𝒯_i)$ (i ∈ I) is a family of NT topologies, $\bigcup_{i \in J} A_i$ ∈ $𝒯_i$. Thus, it is clear that $\bigcup_{i \in J} A_i$ ∈∩$𝒯_i$.

iv) Let A_1, A_2 ∈ ∩ $𝒯_i$. It is clear that A_i ∈ $𝒯_i$. Since $(𝒯_i)$ (i ∈ I) is a family of NT topologies, $A_1 \cap A_2$ ∈ $𝒯_i$. Thus, it is clear that $A_1 \cap A_2$ ∈∩$𝒯_i$.

Definition 3.5: Let $((X, \#), \mathcal{T})$ be a NT topology. For every $A \in \mathcal{T}$, A is called an open set.

Definition 3.6: Let $((X, \#), \mathcal{T})$ be a NT topology space and $A \subset X$. If $(X-A) \in \mathcal{T}$, then A is called close set on X. Where, "X-A" is complement of A according to X.

Corollary 3.7: Let $((X, *), \mathcal{T})$ be a NT topology space. If A is an open set in this space, then X-A is a close set. Also, If B is a close set in this space, then X-B is an open set.

Example 3.8: Let $X = \{x, y, z\}$ be a set, $P(X) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{y, z\}, \{x, z\}, \{x, y, z\}\}$ be power set of X and $\mathcal{T} = \{\emptyset, \{y\}, \{x, y\}, \{y, z\}, \{x, y, z\}\}$. We can take $A \cap A = A$ for $A \in X$. Thus, we can take neut(A)= A, anti(A)=A. Hence, $(P(X), \cap)$ is a NT set. Also, $((X, \cap), \mathcal{T})$ is a NT topology. a) From definition of open set $\emptyset, \{y\}, \{x, y\}, \{y, z\}, \{x, y, z\}$ are open sets.

b) From definition of close set $\{x, y, z\}, \{x, z\}, \{z\}, \{x\}, \emptyset$ are close sets. Because,

$$\{x, y, z\}-\{x, y, z\} = \emptyset \in \mathcal{T}$$

$$\{x, y, z\}-\{x, z\} = \{y\} \in \mathcal{T}$$

$$\{x, y, z\}-\{z\} = \{x, y\} \in \mathcal{T}$$

$$\{x, y, z\}-\{x\} = \{y, z\} \in \mathcal{T}$$

$$\{x, y, z\}-\emptyset = \{x, y, z\} \in \mathcal{T}$$

Theorem 3.9: Let $((X, \#), \mathcal{T})$ be a NT topology space and $\mathcal{K}$ be family of close sets on X. Then,

i) $\emptyset, X \in \mathcal{K}$

ii) For $\forall i \in K$, If $A_i \in X$, then $\bigcap_{i \in K} A_i \in \mathcal{K}$

iii) For $\forall i \in K$, (K is finite) If $A_i \in X$, then $\bigcup_{i \in K} A_i \in \mathcal{K}$

Proof:

i) Since $((X, *), \mathcal{T})$ is a NT topology space, $\emptyset, X \in \mathcal{T}$ and $\emptyset, X$ are open sets. From Corollary 3.7,

$X-X = \emptyset$ is a close set and $X - \emptyset = X$ is a close set. Thus, we obtain $\emptyset, X \in \mathcal{K}$.

ii) Let $B_i \in \mathcal{K}$ (i∈J). Since each B_i is close set, $A_i = X - B_i$ is open set and $B_i = X - A_i$. From definition NT topology, $A = \bigcup_{i \in J} A_i \in \mathcal{T}$. Thus, $F = X- A = X - (\bigcup_{i \in J} A_i) = \bigcap_{i \in J}(X - A_i) = \bigcap_{i \in J} F_i \in \mathcal{K}$.

iii) Let $B_i \in \mathcal{K}$ (i∈J and j is finite). Since each B_i is close set, $A_i = X - B_i$ is open set and $B_i = X - A_i$. From definition NT topology, $A = \bigcap_{i \in J} A_i \in \mathcal{T}$. Thus, $F = X - A = X - (\bigcap_{i \in J} A_i) = \bigcup_{i \in J}(X - A_i) = \bigcup_{i \in J} F_i \in \mathcal{K}$.

Theorem 3.10: Let $((X, *), \mathcal{T})$ be a NT topology space and A be an open set and K be a close set in this space. Then,

i) A - K is an open set.

ii) K – A is a close set.

Proof:

i) It is clear that A-K = A∩(X-K). From Corollary 3.7, X-K is an open set. Thus, from definition of NT topology, A∩(X-K) = A-K is an open set.

ii) It is clear that K-A = K∩(X-A). From Corollary 3.7, X-A is a close set. Thus, from Theorem 3.8, K∩(X-A) = A-K is a close set.

Definition 3.11: Let $((X, *), \mathcal{T_1})$ and $((X, *), \mathcal{T_2})$ be two NT topology spaces and A be an open set according to $\mathcal{T_1}$. For every set A, If A is an open set according to $\mathcal{T_2}$ then, it is called that $\mathcal{T_1}$ is coarser than $\mathcal{T_2}$ or $\mathcal{T_2}$ is called that $\mathcal{T_2}$ is thinner than $\mathcal{T_1}$.

Example 3.12: Let X = {x, y, z} be a set, P(X) = {Ø, {x}, {y}, {z}, {x, y}, {y, z}, {x, z}, {x, y, z}} be power set of X, $\mathcal{T_1}$ = {Ø, {y}, {x, y}, {x, y, z}} and $\mathcal{T_2}$ = {Ø, {y}, {x, y}, {y, z}, {x, y, z}}. From Example 3.2 and Example 3.3, we can take $((X,\cup), \mathcal{T_1})$ and $((X, \cap), \mathcal{T_2})$ are NT topologies. Thus,

Ø, {y}, {x, y}, {x, y, z} are open sets according to $\mathcal{T_1}$,

Ø, {y}, {x, y}, {y, z}, {x, y, z} are open sets according to $\mathcal{T_2}$.

Furthermore, from Definition 3.11, $\mathcal{T_1}$ is coarser than $\mathcal{T_2}$ or $\mathcal{T_2}$ is thinner than $\mathcal{T_1}$.

Definition 3.13: Let $((X, \#), \mathcal{T_1})$ and $((X, *), \mathcal{T_2})$ be two NT topology spaces. If $\mathcal{T_1}$ is coarser than $\mathcal{T_2}$ or $\mathcal{T_2}$ is coarser than $\mathcal{T_1}$, it is called $\mathcal{T_1}$ and $\mathcal{T_2}$ are able to comparison two topologies.

Example 3.14: In Example 3.12, from Definition 3.13, $\mathcal{T_1}$ and $\mathcal{T_2}$ are able to comparison two topologies.

Definition 3.15: Let $((X, \#), \mathcal{T})$ be a NT topology space and x ∈ X. Each open set A in X is called that open neighborhood of x such that x ∈ A.

Example 3.16: From Example 3.8, $((X, \cap), \mathcal{T})$ is a NT topology such that X = {x, y, z}, P(X) = {Ø, {x}, {y}, {z}, {x, y}, {y, z}, {x, z}, {x, y, z}} is power set of X and $\mathcal{T}$ = {Ø, {y}, {x, y}, {y, z}, {x, y, z}}. Also,

{x, y}, {x, y, z} are open neighborhoods of x.

{y}, {x, y}, {y, z}, {x, y, z} are open neighborhoods of y.

{y, z}, {x, y, z} are open neighborhoods of z.

Definition 3.17: Let $((X, \#), \mathcal{T})$ be a NT topology space, $x \in X$, A be an open neighborhood of x and $V \subset X$. If $A \subset V$, then V is called neighborhood of x.

Example 3.18: In Example 3.16, from Definition 3.17, it is clear that

{x, y}, {x, y, z} are neighborhoods of x.

{y}, {x, y}, {y, z}, {x, y, z} are neighborhoods of y.

{y, z}, {x, y, z} are neighborhoods of z.

Definition 3.19: Let $((X, \#), \mathcal{T})$ be a NT topology space and A, B $\subset$ X. If there exists a open set $T \in \mathcal{T}$ such that $A \subset T \subset B$, then B is called B is neighborhood of A.

Example 3.20: From Example 3.3, $((X, \cap), \mathcal{T})$ is a NT topology such that X = {x, y}, P(X) = {∅, {x}, {y}, {x, y }} is power set of X and $\mathcal{T}$ = {∅, {y}, {x, y}}. Also,

For $\emptyset \subset X$, $\emptyset \subset \{y\} \subset \{x, y\}$. Thus, {x, y} is neighborhood of ∅.

For $\{x\} \subset X$, $\{x\} \subset \{x, y\} \subset \{x, y\}$. Thus, {x, y} is neighborhood of {x}.

For $\{y\} \subset X$, $\{y\} \subset \{y\} \subset \{y\}$, {x, y}. Thus, {x, y} and {y} are neighborhoods of {y}.

For $\{x, y\} \subset X$, $\{x, y\} \subset \{x, y\} \subset \{x, y\}$. Thus, {x, y} is neighborhood of {x, y}.

Theorem 3.21: Let $((X, \#), \mathcal{T})$ be a NT topology space and A $\subset$ X. A is an open set $\Leftrightarrow$ for every x $\in$ A, A is a neighborhood of x.

Proof:

($\Rightarrow$) Let A be an open set. For every x $\in$ A, it is clear that x $\in$ A $\subset$ A and A$\in\mathcal{T}$. Thus, from Definition 3.17, A is a neighborhood of x.

($\Leftarrow$) For every x $\in$ A, A is neighborhood of x. Thus, for each x $\in$ A, there exists an open set $T_x \in \mathcal{T}$ such that x $\in T_x \subset$ A. Thus, we obtained

$x \in \cup T_x$ and $A \subset \cup T_x$ (1)

Also, it is clear that

$\cup T_x \subset$ A because $T_x \subset$ A for every x $\in$ A (2)

From (1) and (2), we obtained $A = \cup T_x$. Where, T_x is an open set because $T_x \in \mathcal{T}$. Thus, from Definition 3.1, A is an open set.

Definition 3.22: Let $((X, \#), \mathcal{T})$ be a NT topology space, $A \subset X$, $x \in X$. If A is neighborhood of x, x is called an inner point of A.

Example 3.23: From example 3.16, for $X = \{x, y, z\}$ and $\mathcal{T} = \{\emptyset, \{y\}, \{x, y\}, \{y, z\}, \{x, y, z\}\}$, $((X, \cap), \mathcal{T})$ is a NT topology Also,

$\{x, y\}, \{x, y, z\}$ are neighborhoods of x. Thus, x is an inner point of $\{x, y\}, \{x, y, z\}$.

$\{y\}, \{x, y\}, \{y, z\}, \{x, y, z\}$ are neighborhoods of y. Thus, y is an inner point of $\{y\}, \{x, y\}, \{y, z\}, \{x, y, z\}$.

$\{y, z\}, \{x, y, z\}$ are neighborhoods of z. Thus, z is an inner point of $\{y, z\}, \{x, y, z\}$.

Definition 3.24: Let $((X, \#), \mathcal{T})$ be a NT topology space, $A \subset X$. If B is a set of every inner point x of A, then B is called inner of A. Also, it is shown with A^o.

Example 3.25: In Example 3.20, for $X = \{x, y, z\}$ and $\mathcal{T} = \{\emptyset, \{y\}, \{x, y\}, \{y, z\}, \{x, y, z\}\}$, $((X, \cap), \mathcal{T})$ is a NT topology Also, it is clear that

$\{y\}^o = \{y\}$

$\{x, y\}^o = \{x, y\}$

$\{y, z\}^o = \{y, z\}$

$\{x, y, z\}^o = \{x, y, z\}$

Theorem 3.26: Let $((X, \#), \mathcal{T})$ be a NT topology space, $A \subset X$. Then,

i) $A^o = \cup B \{B \subset X : B \in \mathcal{T} \text{ and } B \subset A\}$

ii) $A^o \subset A$

iii) A^o is an open set.

iv) A is an open set if and only if $A = A^o$.

Proof:

i) Let $C = \cup B \{B \subset X : B \in \mathcal{T} \text{ and } B \subset A\}$ (3)

We show that $A^o = C$. We take $x \in A^o$. From Definition 3.22, there exists an open set such that $x \in B \subset A$. From (3), we obtained $x \in C$. Thus,

$A^o \subset C$ (4)

Then we take $y \in C$. From (1), there exists a $B \in \mathcal{T}$ such that $y \in B \subset A$. Thus, from Definition 3.22, we obtained $y \in A^o$. Thus,

$$C \subset A^o \tag{5}$$

From (4) and (5), we obtained $A^o = C$.

ii) From i), we can take $A^o = \cup B$ $\{B \subset X : B \in \mathcal{T}$ and $B \subset A\}$. Also, it is clear that $\cup B \subset A$. Thus, $A^o \subset A$.

iii) From i), we can take $A^o = \cup B \{B \subset X : B \in \mathcal{T}$ and $B \subset A\}$. Also, B is an open set. Thus, from   Definition 3.1, A^o is an open set.

iv)

($\Rightarrow$) Let A be an open set. Thus, we obtained $A \in \mathcal{T}$. From (1), it is clear that

$$A \subset C = A^o \tag{6}$$

From ii),

$$A^o \subset A \tag{7}$$

Thus, from (6) and (7), we obtained $A^o = A$.

($\Leftarrow$) Let $A^o = A$. From iii), it is clear that A is an open set.

Definition 3.27: Let $((X, \#), \mathcal{T})$ be a NT topology space, $A \subset X$. If x is an inner point of A', then x is called an outside point of A. Also, If B is a set such that every outside point x of A is in B, then B is called outside of A. Also, it is shown with $(X - A)^o$.

Example 3.28: In Example 3.23, for $X = \{x, y, z\}$ and $\mathcal{T} = \{\emptyset, \{y\}, \{x, y\}, \{y, z\}, \{x, y, z\}\}$, $((X, \cap), \mathcal{T})$ is a NT topology Also, it is clear that

x is an outside point of $\{z\}$ and $\emptyset$.

y is an outside point of $\{x, z\}$, $\{z\}$, $\{x\}$ and $\emptyset$.

z is an outside point of $\{x\}$ and $\emptyset$.

Definition 3.29: Let $((X, *), \mathcal{T})$ be a NT topology space, $A \subset X$, $x \in X$. If there exists at least an element of A in every neighborhood of x, x is called closure point of A.

Example 3.30: In Example 3.18, for $X = \{x, y, z\}$ and $\mathcal{T} = \{\emptyset, \{y\}, \{x, y\}, \{y, z\}, \{x, y, z\}\}$, $((X, \cap), \mathcal{T})$ is a NT topology Also,

$\{x, y\}, \{x, y, z\}$ are neighborhoods of x.

$\{y\}, \{x, y\}, \{y, z\}, \{x, y, z\}$ are neighborhoods of y.

{y, z}, {x, y, z} are neighborhoods of z.

Thus,

x is closure point of {x}, {y}, {x, y} and {x, y, z}.

y is closure point of {y}, {x, y} and {x, y, z}.

z is closure point of {z}, {x, z} and {x, y, z}.

Definition 3.31: Let $((X, \#), \mathcal{T})$ be a NT topology space, $A \subset X$. If B is a set of every closure point x of A, then B is called closure of A. Also, it is shown with A^-.

Example 3.32: In Example 3.30, from Definition 3.31,

$\{x\}^- = \{x\}$

$\{y\}^- = \{x, y\}$

$\{z\}^- = \{z\}$

$\{x, y\}^- = \{x, y\}$

$\{x, y, z\}^- = \{x, y, z\}.$

Conclusions

Topology has many different application areas in classical mathematic. Also, NT structures are a new concept in neutrosophy. In this chapter, we introduced NT topology. We gave some properties and definitions for NT topology. Thus, we have added a new structure to NT structure. Furthermore, thanks to NT topology, researchers can obtain new structure and properties. For example, researchers can define NT metric topology, NT group topology, NT algebraic topology.

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Chapter Five
Isomorphism Theorems for Neutrosophic Triplet G - Modules

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Abstract

In this chapter, we introduce neutrosophic triplet cosets for neutrosophic triplet G-module and neutrosophic triplet quotient G-module. Then, we give some definitions and examples for neutrosophic triplet quotient G-module and neutrosophic triplet cosets. Also, we obtain isomorphism theorems for neutrosophic triplet G-modules and we prove isomorphism theorems for neutrosophic triplet G-modules.

Keywords: Neutrosophic triplet, neutrosophic triplet G-module, neutrosophic triplet quotient G-module, neutrosophic triplet isomorphism theorems

1. Introduction

Smarandache introduced neutrosophy in 1980, which studies a lot of scientific fields. In neutrosophy, there are neutrosophic logic, set and probability in [1]. Neutrosophic logic is a generalization of a lot of logics such as fuzzy logic in [2] and intuitionistic fuzzy logic in [3]. Neutrosophic set denoted by (t, i, f) such that "t" is degree of membership, "i" is degree of indeterminacy and "f" is degree of non-membership. Also, a lot of researchers have studied neutrosophic sets in [4-9]. Furthermore, Smarandache *et al.* obtained neutrosophic triplet (NT) in [10] and they introduced NT groups in [11]. For every element "x" in neutrosophic triplet set A, there exist a neutral of "a" and an opposite of "a". Also, neutral of "x" must different from the classical unitary element. Therefore, the NT set is different from the classical set. Furthermore, a NT "x" denoted by <x, neut(x), anti(x)>. Also, many researchers have introduced NT structures in [12-18].

Curties obtained G-module in [19]. G-module is an algebraic structure constructed on classical group and classical vector space. Also, G-module has an important place in the theory of group representation. Also, a lot of researchers studied G-modules. Recently, Fernandez obtained fuzzy G-modules in [20]; Sinho *et al.* obtained isomorphism theory for fuzzy submodules of G-modules in [21]; Şahin *et al.* introduced soft G-modules in [22]; Sharma *et al.* obtained injectivity of intuitionistic fuzzy G-modules in [23]; Smarandache

et al. obtained neutrosophic triplet G-module in [24] and Şahin *et. al* studied isomorphism theorems for soft G-modules in [25].

In this chapter, we obtain NT cosets for NT G-module and NT quotient G-module. Also, we give isomorphism theorems for NT G-module. In section 2; we give definitions of G-module in [19], NT set in [11], NT G-module in [24], and NT G-module homomorphism in [24]. In section 3, we introduce NT cosets for NT G-module and NT quotient G-module. Then, we give some properties and examples for NT quotient G-module. In section 4, we define kernel of a NT G-module homomorphism and we give some properties and examples for kernel of a NT G-module homomorphism. Furthermore, we give the isomorphism theorems for NT G-module and we prove these theorems. In section 5, we give conclusions.

2. Basic and fundamental concepts

Definition 2.1: [19] Let G be a finite group. A vector space V on a field F is called a G- module if for every $m \in G$ and $v \in V$, there exists a product (called the action of G on V) $v.g \in V$ satisfying the following axioms.

a) $v.e_G = v$, $\forall m \in M$ (e_G is unitary element in G)

b) $v.(m.n) = (v.m).n$, $\forall v \in V$; m, n $\in$ G

c) $(f_1 v_1 + f_2 v_2).m = f_1(v_1.m) + f_2(v_2.m)$, $\forall f_1, f_2 \in F$; $v_1, v_2 \in V$; $m \in G$.

Definition 2.2: [11] Let # be a binary operation. (X, #) is a NT set (NTS) such that

i) There must be neutral of "x" such x#neut(x) = neut(x)#x = x, x $\in$ X.

ii) There must be anti of "x" such x#anti(x) = anti(x)#x = neut(x), x $\in$ X.

Furthermore, a NT "x" is showed with (x, neut(x), anti(x)).

Also, neut(x) must different from classical unitary element.

Definition 2.3: [11] Let (X, #) be a NT set. Then, X is called a NT group such that

a) for all x, y $\in$ X, x*y $\in$ X.

b) for all x, y, z $\in$ X, (x*y)*z = x*(y*z)

Definition 2.4: [12] Let (X, &, $) be a NT set with two binary operations & and $. Then (X, &, $) is called NT field (NTF) such that

1. (F, &) is a commutative NT group,

2. (F, $) is a NT group

3. $x\$ (y\&z) = (x\$y) \& (x\$z)$ and $(y\&z)\$x = (y\$x) \& (z\$x)$ forv every x, y, z $\in$ X.

Definition 2.5: [13] Let $(F, \&_1, \$_1)$ be a NTF and let $(V, \&_2, \$_2)$ be a NTS with binary operations $\&_2$ and $\$_2$. If $(V, \&_2, \$_2)$ is satisfied the following conditions, then it is called a NT vector space (NTVS),

1) $x\&_2y \in V$ and $x \$_2 y \in V$; for every x, y $\in$ V

2) $(x\&_2y) \&_2z = x\&_2 (y\&_2z)$; for every x, y, z $\in$ V

3) $x\&_2y = y\&_2x$; for every x, y $\in$ V

4) $(x\&_2y) \$_2m = (x\$_2m) \&_2(y\$_2m)$; for every m$\in$ F and every x, y $\in$ V

5) $(m\&_1n) \$_2x = (m\$_2x) \&_1(n\$_2x)$; for every m, n $\in$ F and every u $\in$ V

6) $(m\$_1n) \$_2x = m\$_1(n\$_2x)$; for every m, n $\in$ F and every x $\in$ V

7) For every x $\in$ V, there exists at least a neut(y) $\in$ F such that

$x \$_2 neut(y) = neut(y) \$_2 x = x$

Definition 2.6: [24] Let (G, *) be a NT group and $(V, *_1, \#_1)$ be a NT vector space on a NT field $(F, *_2, \#_2)$. $(V, *_1, \#_1)$ is a NT G-module such that

  a) v*m $\in$ V, for all m$\in$ G; v $\in$ V

  b) There exists at least an element m $\in$ G such that

for every v $\in$ V, v*neut(m) = neut(m)* v = v;

  c) v*(m*n)=(v*m)*n, for all v $\in$ V; m, n $\in$ G;

  d) $[((f_1\#_1v_1) *_1 (f_2\#_1v_2))]*m = [f_1\#_1(v_1 *m)] *_1 [f_2\#_1(v_2 *m)]$, $\forall$ f_1, f_2 $\in$ F, v_1, $v_2\in$ V; m$\in$ G.

Definition 2.7: [24] Let $(V, *_1, \#_1)$ be a NT G-module. A NT subvector space $(M, *_1, \#_1)$ of $(V, *_1, \#_1)$ is a NT G-submodule if $(M, *_1, \#_1)$ is also a NT G-module.

Definition 2.8: [24] Let $(V, *_1, \#_1)$ and $(V^*, *_3, \#_3)$ be NT G-modules on NT field $(F, *_2, \#_2)$ and (G, *) be a NT group. A mapping g: V$\rightarrow$ V* is a NTG-module homomorphism if

i) g(neut(v))= neut(g(v))

ii) g(anti(v))= anti(g(v))

iii) $g((f_1 \#_1 v_1) *_1 (f_2 \#_1 v_2)) = (f_1 \#_3 g(v_1)) *_3 (f_2 \#_3 g(v_2))$

iv) $g(v*m) = g(v)*m$; $\forall f_1, f_2 \in F$; $m, v_1, v_2 \in V$; $m \in G$.

Also, if g is 1-1, then g is an isomorphism. The NT G-modules $(V, *_1, \#_1)$ and $(V^*, *_3, \#_3)$ are said to be isomorphic if there exists an isomorphism g : NTV→ NTV*. Then it is showed by $V \cong V^*$.

3. Neutrosophic Triplet Quotient G-modules

In this chapter, we show that neutral element of x according to # binary operation with $neut_\#(x)$ and we show that anti element of x according to # binary operation with $anti_\#(x)$.

Definition 3.1: Let $(V, *_1, \#_1)$ be a NT G-module on NT field $(F, *_2, \#_2)$, $(G, *)$ be a NT group and let $(S, *_1, \#_1)$ be a NT G-submodule of $(V, *_1, \#_1)$. The NT cosets of $(S, *_1, \#_1)$ in $(V, *_1, \#_1)$ are denoted by

$x *_1 S$, for all $x \in V$.

Furthermore,

$(x*_1 S) *_1 (y*_1 S) = (x*_1 y) *_1 S$, for all x, y ∈ V.

$\propto \#_1(x*_1 S) = (\propto \#_1 x)*_1 S$, for all y ∈ V and $\propto \in F$.

$m*(x*_1 S) = (x*_1 S)*m = (m*x) *_1 S$, for all x ∈ V, m ∈ G.

Example 3.2: Let G = {∅, {z}, {y}, {z, y}}. We can take that (G, ∩) is a NT group such $neut_\cap(K) = K$ and $anti_\cap(K) = K$. Also, (G, ∪, ∩) is NT field such that

$neut_\cup(K) = K$ and $anti_\cup(K) = K$,

$neut_\cap(K) = K$ and $anti_\cap(K) = K$.

Furthermore, (G, ∪, ∩) is NT vector space on NT field (G, ∪, ∩) such that

$neut_\cup(N) = N$ and $anti_\cup(N) = N$,

$neut_\cap(N) = N$ and $anti_\cap(N) = N$.

Now, we show that (G, ∪, ∩) is NT G-module on NT field (G, ∪, ∩).

For A, B, C, D, E ∈ G;

a) We can take A∪B ∈ G, for all A ∈ G since G is power set of {z, y}.

b) It is clear that there exist $A \in G$ such that $neut_\cap(A) = A \cup neut_\cap(A) = neut_\cap(A) \cup A$, because $(G, \cap)$ is NT group and $neut_\cap(A) = A$.

c) We can take $A \cup (B \cup C) = (A \cup B) \cup C$, for all A, B, C $\in$ G since G is power set of $\{z, y\}$.

d) We can take $[(A \cap B) \cup (C \cap D)] \cup E = [(A \cap B) \cup E] \cup [(C \cap D) \cup E]$ since G is power set of $\{z, y\}$.

Hence, $(G, \cup, \cap)$ is NT G-module on NT field $(G, \cup, \cap)$. Also, $(S, \cup, \cap)$ is a NT G-submodule of $(G, \cup, \cap)$ such that $S = \{\emptyset, \{z\}\}$. Now, we show NT cosets of $(S, \cup, \cap)$.

$\emptyset \cup S = \{\emptyset, \{z\}\}$,

$\{z\} \cup S = \{\{z\}\}$,

$\{y\} \cup S = \{\{y\}, \{y,z\}\}$,

$\{y, z\} \cup S = \{\{y,z\}\}$,

Definition 3.3: Let $(V, *_1, \#_1)$ be a NT G-module on NT field $(F, *_2, \#_2)$ and Let $(S, *_1, \#_1)$ be a NT G-submodule of $(V, *_1, \#_1)$. The NT quotient G-module of $(V, *_1, \#_1)$ is denoted by $(V, *_1, \#_1)/(S, *_1, \#_1)$ such that

$$(V, *_1, \#_1)/(S, *_1, \#_1) = \{x *_1 S : x \in V\}$$

Corollary 3.4: Let $(V, *_1, \#_1)$ be a NT G-module on NT field $(F, *_2, \#_2)$, $(S, *_1, \#_1)$ be a NT G-submodule of $(V, *_1, \#_1)$ and $(V, *_1, \#_1)/(S, *_1, \#_1)$ be NT quotient G-module of $(V, *_1, \#_1)$. From Definition 3.1 and Definition 3.3, $(V, *_1, \#_1)/(S, *_1, \#_1)$ is set of NT cosets of $(S, *_1, \#_1)$ in $(V, *_1, \#_1)$.

Example 3.5: From Example 3.2,

$(G, \cup, \cap)$ is NT G-module on NT field $(G, \cup, \cap)$. Also, $(S, \cup, \cap)$ is a NT G-submodule of $(G, \cup, \cap)$. Also, NT cosets of $(S, \cup, \cap)$ are

$\emptyset \cup S = \{\emptyset, \{z\}\}$,

$\{z\} \cup S = \{\{z\}\}$,

$\{y\} \cup S = \{\{y\}, \{y, z\}\}$,

$\{y, z\} \cup S = \{\{y, z\}\}$. Thus, from Corollary 3.4,

$(G, \cup, \cap)/(S, \cup, \cap) = \{\{\emptyset, \{z\}\}, \{\{z\}\}, \{\{y\}, \{y, z\}\}, \{\{y, z\}\}\}$

Theorem 3.6: Let $(V, *_1, \#_1)$ be a NT G-module on NT field $(F, *_2, \#_2)$, $(G, *)$ be a NT group, $(S, *_1, \#_1)$ be a NT G-submodule of $(V, *_1, \#_1)$ and $(V, *_1, \#_1)/(S, *_1, \#_1)$ be NT quotient G-module of $(V, *_1, \#_1)$. Then, $(V, *_1, \#_1)/(S, *_1, \#_1)$ is a NT G-module on NT field $(F, *_2, \#_2)$.

Proof: From Definition 3.3, we can take

$(V, *_1, \#_1)/(S, *_1, \#_1) = \{x *_1 S : x \in V\}$.

Also, it is clear that $(V, *_1, \#_1)/(S, *_1, \#_1)$ is a NT vector space on $(F, *_2, \#_2)$ with

$(x *_1 S) *_1 (y *_1 S) = (x *_1 y) *_1 S$, for all x, y $\in$ V,

$\propto \#_1 (x *_1 S) = (\propto \#_1 x) *_1 S$, for all y $\in$ V and $\propto \in$ F,

$m*(x *_1 S) = (x *_1 S)*m = (m*x) *_1 S$, for all x $\in$ V, m $\in$ G.

Now, we show that $(V, *_1, \#_1)/(S, *_1, \#_1)$ is a NT G-module on NT field $(F, *_2, \#_2)$.

a) As $(V, *_1, \#_1)$ is a NT G-module, we can take $g*m \in$ V, for all g$\in$ G; m $\in$ V. Thus, we can take $(g*m) *_1 S \in (V, *_1, \#_1)/(S, *_1, \#_1)$

b) As $(V, *_1, \#_1)$ is a NT G-module, there exists at least an element m $\in$ G such for every v$\in$V,

$v*\text{neut}(m) = \text{neut}(m)* v = v$.

Thus, $v*\text{neut}(m) = \text{neut}(m)*v = v \in$V and we can take

$(v *_1 S)*\text{neut}(m) = \text{neut}(m)* (v *_1 S) = (v *_1 S)$.

c) As $(V, *_1, \#_1)$ is a NT G-module, we can take $v*(n*m)=(v*n)*m$, $\forall$ v $\in$ V; n, m $\in$ G. Thus, $(v *_1 S)*(n*m)=[(v *_1 S)*n]*m$

d) As $(V, *_1, \#_1)$ is a NT G-module, we can take

$[(f_1 \#_1 v_1) *_1 (f_2 \#_1 v_2)]*m = [f_1 \#_1 (v_1 *m)] *_1 [f_2 \#_1 (v_2 *m)]$, $\forall$ f_1, $f_2 \in$ F; v_1, $v_2 \in$V; m$\in$ G. Thus,

$[(f_1 \#_1 (v_1 *_1 S)) *_1 (f_2 \#_1 (v_2 *_1 S))]*m = [f_1 \#_1 ((v_1 *_1 S) *m] *_1 [f_2 \#_1 ((v_2 *_1 S) *m]$ since $(v_1 *_1 S)$ and $(v_2 *_1 S)$ are NT cosets.

Theorem 3.7: Let $(V, *_1, \#_1)$ be a NT G-module on NT field $(F, *_2, \#_2)$, $(V_1, *_1, \#_1)$ and $(V_2, *_1, \#_1)$ be NT G-submodules of $(V, *_1, \#_1)$ and $(G, *)$ be a NT group. Then,

$(V_1, *_1, \#_1) \cap (V_2, *_1, \#_1) = (V_1 \cap V_2, *_1, \#_1)$ is a NT G-submodule of $(V, *_1, \#_1)$.

Proof:

a) If $v \in V_1 \cap V_2$, then $v \in V_1$ and $v \in V_2$. Thus, $v*m \in V_1$ and $v*m \in V_2$ since $(V_1,*_1, \#_1)$ and $(V_2,*_1, \#_1)$ are NT G-submodules of $(V,*_1, \#_1)$. Thus, we obtain $v*m \in V_1 \cap V_2$, for all $m \in G$; $v \in V$.

Also, $(V_1 \cap V_2,*_1, \#_1)$ satisfies the conditions b, c and d, since $V_1 \subset V$ and $V_2 \subset V$ and $(V,*_1, \#_1)$ is a NT G-module.

Theorem 3.8: Let $(V,*_1, \#_1)$ be a NT G-module on NT field $(F, *_2, \#_2)$, $(V_1,*_1, \#_1)$ and $(V_2 ,*_1 , \#_1)$ be NT G-submodules of $(V,*_1 , \#_1)$ and $(G, *)$ be a NT group. If $(v_1*m) *_1 (v_2*m) = (v_1 *_1 v_2)*m$ for $v_1 \in V_1$, $v_2 \in V_2$ and $m \in G$ then,

$(V_1,*_1, \#_1) *_1 (V_2,*_1, \#_1)$ is a NT G-submodule of $(V,*_1, \#_1)$ such that

$v = v_1 *_1 v_2 \in (V_1,*_1, \#_1) *_1 (V_2,*_1, \#_1)$, for all $v_1 \in V_1$ and $v_2 \in V_2$.

Proof:

a) If $v \in (V_1,*_1, \#_1) *_1 (V_2,*_1, \#_1)$, then $v = v_1 *_1 v_2$ for $v_1 \in V_1$ and $v_2 \in V_2$. Thus, $v_1*m \in V_1$ and $v_2*m \in V_2$ for all $m \in G$ since $(V_1,*_1, \#_1)$ and $(V_2,*_1, \#_1)$ be NT G-submodules of $(V,*_1, \#_1)$. Also, $(v_1*m) *_1 (v_2*m) \in (V_1,*_1, \#_1) *_1 (V_2,*_1, \#_1)$. Thus, we obtain $(v_1 *_1 v_2)*m = v*m \in (V_1,*_1, \#_1) *_1 (V_2,*_1, \#_1)$

since $(v_1*m) *_1 (v_2*m) = (v_1 *_1 v_2)*m$ for $v_1 \in V_1$, $v_2 \in V_2$ and $m \in G$.

Also, $(V_1,*_1, \#_1) *_1 (V_2,*_1, \#_1)$ satisfies the conditions b, c and d, since $V_1 \subset V$ and $V_2 \subset V$ and $(V,*_1, \#_1)$ is a NT G-module.

Theorem 3.9: Let $(V,*_1, \#_1)$ be a NT G-module on NT field $(F, *_2, \#_2)$, $(V_1,*_1, \#_1)$ be NT G-submodule of $(V,*_1, \#_1)$, $(V_2,*_1, \#_1)$ be NT G-submodule of $(V_1,*_1, \#_1)$ and $(G, *)$ be a NT group. Then,

$(V_1,*_1, \#_1)/ (V_2,*_1, \#_1)$ is a NT G-submodule of $(V,*_1, \#_1)/ (V_2,*_1, \#_1)$.

Proof: It is clear that $V_1/V_2 \subset V/V_2$ since $(V_1,*_1, \#_1)$ is a NT G-submodule of $(V,*_1, \#_1)$. Then, for $x \in (V_1,*_1, \#_1)/ (V_2,*_1, \#_1)$, $x = v*_1(V_2,*_1, \#_1)$.Where, $v \in V_1$. Also, it is clear that $v*m \in V$ since $(V_1,*_1, \#_1)$ is NT G – submodule for $m \in G$. Also,

$x*m = (v*_1(V_2,*_1, \#_1))*m = (v*m)*_1(V_2,*_1, \#_1))$ since $(V_1,*_1, \#_1)/ (V_2,*_1, \#_1)$ is a NT quotient G-module. Thus, $x*m \in (V_1,*_1, \#_1)/ (V_2,*_1, \#_1)$.

Also, $(V_1,*_1, \#_1)/(V_2,*_1, \#_1)$ satisfies conditions b, c, and d since $V_1/V_2 \subset V/V_2$ and $(V,*_1, \#_1)/ (V_2,*_1, \#_1)$ is a NT quotient G-module.

4. Isomorphism Theorems for Neutrosophic Triplet G-modules

Definition 4.1: Let $(V_1, *_1, \#_1)$ and Let $(V_2, *_3, \#_3)$ be NT G-modules on NT field $(F, *_2, \#_2)$, g be a NT G-module homomorphism such that g: $V_1 \to V_2$. Then, kernel of g is a set such that $\{x \in V_1 : g(x) = neut_{*_3}(y), y \in V_2\}$ and it is denoted by kerg.

Example 4.2: From Example 3.2, for G = $\{\emptyset, \{z\}, \{y\}, \{z, y\}\}$, $(G, \cap)$ is a NT group $(G, \cup, \cap)$ is NT G-module on NT field $(G, \cup, \cap)$. Where,

$neut_\cup(N) = N$ and $anti_\cup(N) = N$,

$neut_\cap(N) = N$ and $anti_\cap(N) = N$.

Then, we take g: $G \to G$ mapping such that $g(A) = anti_\cup(A)$. For all A, B, C, D $\in$ G,

i) $g(neut_\cup(A)) = anti_\cup(neut_\cup(A))$. From Theorem 2.4, $g(neut_\cup(A)) = anti_\cup(neut_\cup(A)) = neut_\cup(A)$. Also, $g(neut_\cup(A)) = neut_\cup(g(A))$ since $neut_\cup(A) = A = anti_\cup(N)$.

ii) $g(anti_\cup(A)) = anti_\cup(anti_\cup(A))$. From Theorem 2.4, $g(anti_\cup(A)) = anti_\cup(anti_\cup(A)) = A$. Also, $g(anti_\cup(A)) = anti_\cup(g(A))$ since $neut_\cup(A) = A = anti_\cup(N)$

iii) $g((A \cap B) \cup (C \cap D)) = anti_\cup((A \cap B) \cup (C \cap D)) = ((A \cap B) \cup (C \cap D)) = ((A \cap anti_\cup(B)) \cup (C \cap anti_\cup(D))) = ((A \cap g(B)) \cup (C \cap g(D)))$ since $A = anti_\cup(A)$.

iv) $g(A \cap B) = anti_\cup(A \cap B) = A \cap B = anti_\cup(A) \cap B = g(A) \cap B$ since $A = anti_\cup(A)$.

Thus, $g(A) = anti_\cup(A)$ is a NT G-module homomorphism. Also, $g(A) = anti_\cup(A)$ is a 1-1 NT G-module since $neut_\cup(A) = A = anti_\cup(N)$. Therefore, $g(A) = anti_\cup(A)$ is a NT G-module isomorphism.

Also, from Definition 4.1, kerg = $\{A \in G : g(A) = neut_\cup(A)\} = G$, since $neut_\cup(A) = A = anti_\cup(N)$.

Theorem 4.3: Let $(V_1, *_1, \#_1)$ and Let $(V_2, *_3, \#_3)$ be NT G-modules on NT field $(F, *_2, \#_2)$, g be a NT G-module homomorphism such that g: $V_1 \to V_2$ and $(G, *)$ be a NT group. Then,

i) kerg $\subset V_1$

ii) If $neut_{*_3}(x) * m = neut_{*_3}(x * m)$, then (kerg, $*_1, \#_1$) is a NT G-submodule of $(V_1, *_1, \#_1)$, $x \in V_1$, $m \in G$.

Proof:

i) It is clear that kerg $\subset V_1$ since kerg = $\{x \in V_1 : g(x) = neut_{*_3}(y), y \in V_2\}$.

ii) Let $m \in G$, $x \in$ kerg and $neut_{*_1}(x) * m = neut_{*_1}(x*m)$. Now, we show that (kerg, $*_1$, $\#_1$) is a NT G-submodule of $(V_1, *_1, \#_1)$.

a) From Definition 2.9, $g(x*m) = g(x)*m$ since g is a NT G-module homomorphism. Also, for y $y \in V_2$, $g(x*m) = g(x)*m = neut_{*_3}(y)*m$, if $x \in$ kerg. Furthermore,

$g(x*m) = g(x)*m = neut_{*_3}(y)*m = neut_{*_3}(y*m)$, since $neut_{*_3}(y)*m = neut_{*_3}(y*m)$. Also, $y*m \in V_2$ since $(V_2, *_3, \#_3)$ is a NT G-modules. Thus, $x*m \in$ kerg.

Also, (kerg, $*_1$, $\#_1$) is satisfied the conditions b, c, d in Definition 2.7, since kerg$\subset V_1$. Therefore, (kerg, $*_1$, $\#_1$) is a NT G-submodule of $(V_1, *_1, \#_1)$.

Theorem 4.4: Let $(V_1, *_1, \#_1)$ and Let $(V_2, *_3, \#_3)$ be NT G-modules on NT field $(F, *_2, \#_2)$, g be a NT G-module homomorphism such that g: $V_1 \rightarrow V_2$ and (G, *) be a NT group. If $neut_{*_1}(x)*m = neut_{*_1}(x*m)$ for all $x \in V_1$, $m \in G$ then, there exists a

f: $(V_1, *_1, \#_1)$/kerg$\rightarrow (V_2, *_3, \#_3)$ mapping such that f is a NT homomorphism.

Proof: Let f: V_1/kerg$\rightarrow V_2$ be a mapping such that $f(x*_1$kerg$) = g(x)$.

i) We can take $f(neut_{*_1}(x *_1$ kerg$)) = f(neut_{*_1}(x) *_1$ kerg$)$ since $neut_{*_1}(x) * m = neut_{*_1}(x*m)$. Also,

$f(neut_{*_1}(x *_1$ kerg$)) = f(neut_{*_1}(x) *_1$ kerg$) = g(neut_{*_1}(x)) = neut_{*_1}g(x) = neut_{*_1}f(x*_1$kerg$)$ since g is a NT homomorphism.

ii) We can take $f(anti_{*_1}(x*_1$kerg$)) = f(anti_{*_1}(x) *_1$kerg$)$ since (kerg, $*_1$, $\#_1$) is a NT G-submodule of $(V_1, *_1, \#_1)$. Also,

$f(anti_{*_1}(x*_1$kerg$)) = f(anti_{*_1}(x) *_1$kerg$) = g(anti_{*_1}(x)) = anti_{*_1}g(x) = anti_{*_1}f(x*_1$kerg$)$ since g is a NT homomorphism.

iii) We can take $g((k_1\#_1(m_1) *_1 (k_2 \#_1(m_2)) = (k_1\#_3g(m_1)) *_3(k_2\#_3g(m_2))$ since g is a NT homomorphism. Also,

$g((k_1\#_1(m_1) *_1 (k_2 \#_1(m_2)) = f([(k_1\#_1(m_1) *_1 (k_2 \#_1(m_2)] *_1$kerg$] =$

$f([(k_1\#_1(m_1 *_1$ kerg$) *_1 (k_2 \#_1(m_2 *_1$ kerg$)] =$

$(k_1\#_3g(m_1)) *_3(k_2\#_3g(m_2)) =$

$(k_1\#_3f(m_1 *_1$ kerg$)) *_3(k_2\#_3f(m_2 *_1$ kerg$))$

since $f(x*_1$kerg$) = g(x)$ and $m*_1$kerg is NT cosets of $(V_1, *_1, \#_1)$.

iv) We can take g(m*n) = g(m)*n since g is a NT homomorphism. Also,

g(m*n) = f((m*n)*$_1$kerg)) = f((m*$_1$kerg)*n) = g(m)*n = f(m*$_1$kerg)*n since m*$_1$kerg is NT cosets of (V_1,*$_1$, #$_1$) and g is a NT homomorphism. Thus, f is a NT G-module homomorphism.

Corollary 4.5: Let (V_1,*$_1$, #$_1$) and Let (V_2,*$_3$, #$_3$) be NT G-modules on NT field (F, *$_2$, #$_2$), g be a NT   G-module homomorphism such that g: $V_1 \rightarrow V_2$ and (G, *) be a NT group. If $neut_{*_1}$(x)*m = $neut_{*_1}$(x*m) for all x∈ V_1, m∈ G, g is 1-1 and surjection, then there exists a

f: (V_1,*$_1$, #$_1$)/kerg$\rightarrow$ (V_2,*$_3$, #$_3$) mapping such that V_1/kerg $\cong V_2$.

Proof: If we take that f: V_1/kerg$\rightarrow V_2$ is a mapping such that f(x*$_1$kerg) = g(x), then from Theorem 4.4;   f is a NT G-module homomorphism. Also, we assume that

f(x*$_1$kerg) = f(y*$_1$kerg). Thus, f(x*$_1$kerg) = g(x) = g(y) = f(y*$_1$kerg). Also,   x = y since f is 1-1. Therefore, f is 1-1.

Also, it is clear that f is surjection since g is surjection.

Theorem 4.6: Let (V,*$_1$, #$_1$) be a NT G-module on NT field (F, *$_2$, #$_2$), (V_1,*$_1$, #$_1$) be NT G-submodule of (V,*$_1$, #$_1$), (V_2,*$_1$, #$_1$) be NT G-submodule of (V_1,*$_1$, #$_1$) and (G, *) be a NT group. Then, there exists a

f: (V,*$_1$, #$_1$)/(V_1,*$_1$, #$_1$)$\rightarrow$ [(V,*$_1$, #$_1$)/ (V_2,*$_1$, #$_1$)]/ [(V_1,*$_1$, #$_1$)/ (V_2,*$_1$, #$_1$)]

mapping such that f is a NT homomorphism.

Proof: from Theorem 3.9, it is clear that (V_1,*$_1$, #$_1$)/ (V_2,*$_1$, #$_1$) is a NT G-submodule of (V,*$_1$, #$_1$)/ (V_2,*$_1$, #$_1$)

Also, let f: V/$V_1 \rightarrow$ (V/V_2) / (V_1/V_2) be a mapping such that

f(x*$_1$ V_1) = (x*$_1$ V_2) *$_1$(V_1/V_2)).

i) We can take f($neut_{*_1}$(x) *$_1$ V_1) = $neut_{*_1}$(x*$_1$ V_2) *$_1$(V_1/V_2)) = $neut_{*_1}$ f(x*$_1$ V_1) since (V_2 *$_1$(V_1/V_2)) is a NT quotient G-module.

ii) We can take f($anti_{*_1}$x*$_1$ V_1) = $anti_{*_1}$(x*$_1$ V_2) *$_1$ (V_1/V_2)) = $anti_{*_1}$ f(x*$_1$ V_1) since (V_2 *$_1$(V_1/V_2)) is a NT quotient G-module.

iii) We can take

 f([(k_1#$_1$(m_1) *$_1$ (k_2 #$_1$(m_2)] *$_1$ V_1) =([k_1#$_1$(m_1) *$_1$ (k_2 #$_1$(m_2)] *$_1$ V_2) *$_1$(V_1/V_2)) =

$([k_1 \#_1 (m_1 *_1 V_2) *_1 (V_1/V_2))] *_1 [k_2 \#_1 (m_2 *_1 V_2) *_1 (V_1/V_2))]$, since $(V_2 *_1 (V_1/V_2))$ is a NT quotient G-module. Thus,

$f((k_1 \#_1 (m_1) *_1 (k_2 \#_1 (m_2) *_1 V_1) = ([k_1 \#_1 (m_1) *_1 (k_2 \#_1 (m_2)] *_1 V_2) *_1 (V_1/V_2)) =$

$[k_1 \#_1 (m_1 *_1 V_2) *_1 (V_1/V_2))] *_1 [k_2 \#_1 (m_2 *_1 V_2) *_1 (V_1/V_2))] =$

$(k_1 \#_1 f(m_1)) *_1 (k_2 \#_1 f(m_2))$.

iv) We can take $f((x*g) *_1 V_1) = ((x*g) *_1 V_2) *_1 (V_1/V_2)) = [x *_1 V_2) *_1 (V_1/V_2))]*g = f(x)*g$, since $(V_2 *_1 (V_1/V_2))$ is a NT quotient G-module. Thus, f is a NT G-module homomorphism.

Corollary 4.7: From Theorem 4.6, Let $(V, *_1, \#_1)$ be a NT G-module on NT field $(F, *_2, \#_2)$, $(V_1, *_1, \#_1)$ be NT G-submodule of $(V, *_1, \#_1)$, $(V_2, *_1, \#_1)$ be NT G-submodule of $(V_1, *_1, \#_1)$ and $(G, *)$ be a NT group. If there exists a

$f: (V, *_1, \#_1)/(V_1, *_1, \#_1) \rightarrow [(V, *_1, \#_1)/ (V_2, *_1, \#_1)]/ [(V_1, *_1, \#_1)/ (V_2, *_1, \#_1)]$

mapping such that f is 1-1 and surjection, then

$(V, *_1, \#_1)/(V_1, *_1, \#_1) \cong [(V, *_1, \#_1)/ (V_2, *_1, \#_1)]/ [(V_1, *_1, \#_1)/ (V_2, *_1, \#_1)]$.

Theorem 4.8: Let $(V, *_1, \#_1)$ be a NT G-module on NT field $(F, *_2, \#_2)$, $(V_1, *_1, \#_1)$ and $(V_2, *_1, \#_1)$ be NT G-submodules of $(V, *_1, \#_1)$ and $(G, *)$ be a NT group. If $(v_1 * m) *_1 (v_2 * m) = (v_1 *_1 v_2)*m$ for $v_1 \in V_1$, $v_2 \in V_2$ and $m \in G$ then, there exists a

$f: (V, *_1, \#_1)/[(V_1, *_1, \#_1) \cap (V_2, *_1, \#_1)] \rightarrow (V, *_1, \#_1) /[(V_1, *_1, \#_1) *_1 (V_2, *_1, \#_1)]$

mapping such that f is a NT homomorphism.

Proof:

From Theorem 3.7, $(V_1, *_1, \#_1) \cap (V_2, *_1, \#_1)$ is a NT G- submodule of $(V, *_1, \#_1)$. Also, from Theorem 3.8, $(V_1, *_1, \#_1) *_1 (V_2, *_1, \#_1)$ is a NT G- submodule of $(V, *_1, \#_1)$ since $(v_1 * m) *_1 (v_2 * m) = (v_1 *_1 v_2)*m$ for $v_1 \in V_1$, $v_2 \in V_2$ and $m \in G$.

Also, let $f: V/(V_1 \cap V_2) \rightarrow V/(V_1 *_1 V_2)$ be a mapping such that

$f(x *_1 (V_1 \cap V_2)) = x *_1 (V_1 *_1 V_2)$, for $x \in V$.

i) We can take

$f(neut_{*_1} [x *_1 (V_1 \cap V_2)]) = neut_{*_1} x(*_1 (V_1 *_1 V_2)) = neut_{*_1} f(x *_1 (V_1 \cap V_2))$ since $V/(V_1 *_1 V_2)$ is a NT quotient G-module.

ii) We can take f($anti_{*_1}$ [x$*_1$ ($V_1 \cap V_2$)]) = $anti_{*_1}$(x$*_1$ ($V_1 *_1 V_2$)) = $anti_{*_1}$f(x$*_1$ ($V_1 \cap V_2$)) since V/($V_1 *_1 V_2$) is a NT quotient G-module.

iii) We can take

f([($k_1\#_1(m_1)$) $*_1$ ($k_2 \#_1(m_2)$)] $*_1$ ($V_1 \cap V_2$)) =([$k_1\#_1(m_1)$ $*_1$ ($k_2 \#_1(m_2)$)] $*_1$ V_2)=

([$k_1\#_1(m_1 *_1$ ($V_1 *_1 V_2$))] $*_1$ [$k_2\#_1(m_2 *_1 (V_1 *_1 V_2$))] , since $V/(V *_1 V_2)$ is a NT quotient G-module. Thus,

f([($k_1\#_1(m_1)$) $*_1$ ($k_2 \#_1(m_2)$)] $*_1$ ($V_1 \cap V_2$)) =([$k_1\#_1(m_1)$ $*_1$ ($k_2 \#_1(m_2)$)] $*_1$ ($V_1 *_1 V_2$))=

([$k_1\#_1(m_1 *_1 (V_1 *_1 V_2$))] $*_1$ [$k_2\#_1(m_2 *_1 (V_1 *_1 V_2$))] =($k_1\#_1$f($m_1$)) $*_1$ ($k_2\#_1$f(m_2)).

iv) We can take f((x*g) $*_1$ ($V_1 \cap V_2$)) = (x*g)$*_1$ ($V_1 *_1 V_2$))= (x$*_1$ ($V_1 *_1 V_2$))*g = f(x)*g, since V/($V_1 *_1 V_2$) is a NT quotient G-module. Thus, f is a NT G-module homomorphism.

Corollary 4.9: From Theorem 4.8, Let (V,$*_1$, $\#_1$) be a NT G-module on NT field (F, $*_2$, $\#_2$), (V_1,$*_1$, $\#_1$) and (V_2,$*_1$, $\#_1$) be NT G-submodules of (V,$*_1$, $\#_1$) and (G, *) be a NT group. If there exists a

f: (V,$*_1$, $\#_1$)/[(V_1,$*_1$, $\#_1$)$\cap$ (V_2,$*_1$, $\#_1$)]$\rightarrow$ (V,$*_1$, $\#_1$) /[(V_1,$*_1$, $\#_1$) $*_1$(V_2,$*_1$, $\#_1$)]

mapping such that f is1-1 and surjection, then

 (V,$*_1$, $\#_1$)/[(V_1,$*_1$, $\#_1$)$\cap$ (V_2,$*_1$, $\#_1$)]$\cong$ (V,$*_1$, $\#_1$) /[(V_1,$*_1$, $\#_1$) $*_1$(V_2,$*_1$, $\#_1$)].

Conclusions

In this chapter, we obtained NT cosets for NT G-modules and NT quotient G-module. Also, we gave isomorphism theorems for NT G-modules and we proved these theorems. Thus, we have added a new structure to NT structure and we gave a rise to a new field or research called NTPIPS. Also, thanks to NT cosets for NT G-modules and NT quotient G-module researchers can obtain new structures and properties.

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Chapter Six
Neutrosophic Triplet Lie Algebra

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Abstract

In this chapter, we firstly introduce neutrosophic triplet lie algebra. Furthermore, we give some definitions and examples for neutrosophic triplet lie algebra. Then, we obtain that neutrosophic triplet lie algebra is different from classical lie algebra.

Keywords: neutrosophic triplet set, lie algebra, neutrosophic triplet lie algebra

1. Introduction

Smarandache introduced neutrosophy in 1980, which studies a lot of scientific fields. In neutrosophy, there are neutrosophic logic, set and probability in [1]. Neutrosophic logic is a generalization of a lot of logics such as fuzzy logic in [2] and intuitionistic fuzzy logic in [3]. Neutrosophic set denoted by (t, i, f) such that "t" is degree of membership, "i" is degree of indeterminacy and "f" is degree of non-membership. Also, a lot of researchers have studied neutrosophic sets in [4-9]. Furthermore, Smarandache *et al.* obtained neutrosophic triplet (NT) in [10] and they introduced NT groups in [11]. For every element "x" in neutrosophic triplet set A, there exist a neutral of "a" and an opposite of "a". Also, neutral of "x" must different from the classical unitary element. Therefore, the NT set is different from the classical set. Furthermore, a NT "x" denoted by <x, neut(x), anti(x)>. Also, many researchers have introduced NT structures in [12-19].

Sophus Lie introduced lie theory. Since the twentieth century, lie algebras have been used in many fields, particularly in topology, algebra, differential geometry, representation theory, harmonic analysis and mathematical physics. Also, many researches have done many studies related to lie algebra. Recently, Erdmann *et al.* studied lie algebras in [20], Cahn introduced semi-simple lie algebras and their representations in [21], Yehia obtained fuzzy ideals and fuzzy subalgebras of lie algebras in [22], Akram studied fuzzy soft lie algebras in [23].

In this chapter, we obtain NT lie algebras and we give some definitions and examples for neutrosophic triplet lie algebra In section 2; we give definitions lie algebra in [20], NT set in [11], NT group in [11], NT field in [12], and NT vector space in [13]. In section 3, we introduce NT lie algebras and examples for NT lie algebra. We show that NT lie algebras

different from classical lie algebras. Also, we show relationship between NT lie algebra and classical lie algebra. Furthermore, we define NT lie algebra homomorphism, NT lie subalgebra, NT lie coset and NT lie quotient algebras and we give examples for those structures. In section 4, we give conclusions.

2. Basic and fundamental concepts

Definition 2.1: [19] Let F be a field and M be a vector space on F. Then the mapping [.,.]:MxM→M is called lie algebra on M such that

i) $[x+\alpha y, z] = [x, z] + \alpha[y, z]$ and $[x, y+\beta z] = [x, y] + \beta[x, z]$; x, y, z ∈ M and α, β ∈ F (bilinear mapping)

ii) [x, x] = 0

iii) [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 or [x, y], z]] + [y, z], x]] + [z, x], y]] = 0

Definition 2.2: [11] Let # be a binary operation. (X, #) is a NTS such that

i) There must be neutral of "x" such x#neut(x) = neut(x)#x = x, x ∈ X.

ii) There must be anti of "x" such x#anti(x) = anti(x)#x = neut(x), x ∈ X.

Furthermore, a NT "x" is showed with (x, neut(x), anti(x)).

Also, neut(x) must different from classical unitary element.

Definition 2.3: [11] Let (X, #) be a NTS. Then, X is called a NTG such that

a) For all x, y ∈ X, x*y ∈ X.

b) For all x, y, z ∈ X, (x*y)*z = x*(y*z)

Definition 2.4: [12] Let (X, &, $) be a NTS with two binary operations & and $. Then, (X, &, $) is called NTF such that

1. (F, &) is a commutative NTG,

2. (F, $) is a NTG,

3. x$(y&z) = (x$y) & (x$z) and (y&z)$x = (y$x) & (z$x) forv every x, y, z ∈ X.

Definition 2.5: [13] Let $(F, \&_1, \$_1)$ be a NTF and let $(V, \&_2, \$_2)$ be a NTS with binary operations "$\&_2$" and "$\$_2$". If $(V, \&_2, \$_2)$ is satisfied the following conditions, then it is called a NTVS,

1) $x\&_2 y$ ∈ V and x $\$_2 y$ ∈ V; for every x, y ∈ V

2) $(x\&_2 y)\&_2 z = x\&_2 (y\&_2 z)$; for every x, y, z ∈ V

3) $x \&_2 y = y \&_2 x$; for every x, y $\in$ V

4) $(x \&_2 y) \$_2 m = (x \$_2 m) \&_2 (y \$_2 m)$; for every m$\in$ F and every x, y $\in$ V

5) $(m \&_1 n) \$_2 x = (m \$_2 x) \&_1 (n \$_2 x)$; for every m, n $\in$ F and every u $\in$ V

6) $(m \$_1 n) \$_2 x = m \$_1 (n \$_2 x)$; for every m, n $\in$ F and every x $\in$ V

7) For every x $\in$ V, there exists at least a neut(y) $\in$ F such that

 x $\$_2$ neut(y)= neut(y) $\$_2$ x = x

3. Neutrosophic Triplet Lie Algebra

In this paper, we show that neutral element of x according to # binary operation with $neut_\#(x)$ and we show that anti element of x according to # binary operation with $anti_\#(x)$.

Definition 3.1: Let $(V, *_1, \#_1)$ be a NTVS on NTF $(F, *_2, \#_2)$. Then the mapping [.,.]:VxV→V is called a NT lie algebra on $(V, *_1, \#_1)$ such that

i) $[x *_1 (\alpha \#_1 y), z] = [x, z] *_1 (\alpha \#_1 [y, z])$ and

$[x, y *_1 (\beta \#_1 z)] = [x, y] *_1 (\beta \#_1 [x, z])$; $\forall$ x, y, z $\in$ V and $\alpha, \beta \in$ F (bilinear mapping)

ii) There exists at least an element t $= neut_{*_1}(t) \in$ V such that $[x, x] = neut_{*_1}(t)$, for each x $\in$ V.

iii)   There   exists   at   least   an   element   t   = $neut_{*_1}$ (t) $\in$ V   such   that
      $[x, [y, z]] *_1 [y, [z, x]] *_1 [z, [x, y]] = neut_{*_1}(t)$, for each x, y, z $\in$ V triplet;

or

      $[x, y], z]] *_1 [y, z], x]] *_1 [z, x], y]] = neut_{*_1}(t)$ for each x, y, z $\in$ V triplet;

Example 3.2: Let V = {∅, {a}, {b}, {a, b}}. We can take that (V, ∪, ∩) is NTF such that

      $neut_\cup(X) = X$,

      $anti_\cup(X) = Y$, Y$\subset$X   (1)

and

      $neut_\cap(X) = X$,

      $anti_\cap(X) = Y$,   X$\subset$Y   (2)

Also, (V, ∪, ∩) is NTVS on NTF (V, ∪, ∩) with (1) and (2).

We take the mapping [.,.]:VxV→V such that [A, B] = A∪B. Now show that [A, B] = A∪B is a NT lie algebra.

i) a)

It is clear that if A = C = D =B or B = Ø, then It is clear that

[A∪ (B ∩ C), D] = [A, D] ∪ (B ∩[C, D]). Also,

[Ø ∪ ({a, b} ∩ {a}), {b}] = {a, b} = [Ø,{b}] ∪ ({a, b} ∩[{a}, {b}]).

[Ø ∪ ({a} ∩ {a}), {b}] = {a, b} = [Ø,{b}] ∪ ({a} ∩[{a}, {b}]).

[Ø ∪ ({ b} ∩ {a}), {b}] = { b} = [Ø,{b}] ∪ ({b} ∩[{a}, {b}]).

[{a} ∪ ({a, b} ∩ {a}), {b}] = {a, b} = [{a},{b}] ∪ ({a, b} ∩[{a}, {b}]).

[{b} ∪ ({a, b} ∩ {a}), {b}] = {a, b} = [{b},{b}] ∪ ({a, b} ∩[{a}, {b}]).

[{a, b} ∪ ({a, b} ∩ {a}), {b}] = {a, b} = [{a, b},{b}] ∪ ({a, b} ∩[{a}, {b}]).

[Ø ∪ ({a, b} ∩ {a}), Ø] = {a} = [Ø,Ø ∪ ({a, b} ∩[{a}, Ø]).

[{a} ∪ ({a, b} ∩ {a}), Ø] = {a} = [{a},Ø] ∪ ({a, b} ∩[{a}, Ø]).

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b) It can be show similarly to a).

ii) [A, A] = A∪A = A = $neut_A$(A). So, there exists an element A = $neut_∪$(A) ∈ V such that [A, A] = $neut_∪$(A), for each A ∈ V.

iii) [A, [B, C]] ∪ [B, [C, A]] ∪ [C, [A, B]] = A∪B∪C = $neut_∪$(A∪B∪C). So, there exists an element A∪B∪C = $neut_∪$(A∪B∪C) ∈ V such that

[A, [B, C]] ∪ [B, [C, A]] ∪ [C, [A, B]] = $neut_∪$(A∪B∪C), for each A, B, C ∈ V triplet. Similarly, there exists an element A∪B∪C = $neut_∪$(A∪B∪C) ∈ V such that

[A, B], C]] ∪ [B, C], A]] ∪ [C, A], B]] = $neut_∪$(A∪B∪C), for each A, B, C ∈ V triplet.

Therefore, [A, B] = A∪B is a NT lie algebra on (V, ∪, ∩).

Corollary 3.3: From Definition 3.1 and Definition 2.1, it is clear that NT lie algebra is different from classical lie algebra. Because, $neut_{*_1}(t)$ is different from classical unitary element and $neut_{*_1}(t)$ can be more than one.

Corollary 3.4: Let $(V, *_1, \#_1)$ be a NTVS on NTF $(F, *_2, \#_2)$ and the mapping $[.,.]:VxV{\rightarrow}V$ be a NT lie algebra on $(V, *_1, \#_1)$. In Definition 3.1, if we take classical vector space instead of $(V, *_1, \#_1)$ and we take classical field instead of $(F, *_2, \#_2)$, then the mapping $[.,.]$ NT lie algebra satisfies the classical lie algebra's conditions.

Proof: In classical vector space, it is clear that classical unitary element must be one. So, in Definition 3.1, $neut_{*_1}(t)$ must be one and $neut_{*_1}(t)$ must be equal to classical unitary element 0. Therefore, $[.,.]$ NT lie algebra satisfies conditions in Definition 2.1.

Theorem 3.5: Let $(V, *_1, \#_1)$ be a NTVS on NTF $(F, *_2, \#_2)$, the mapping $[.,.]:VxV{\rightarrow}V$ be a NT lie algebra on $(V, *_1, \#_1)$ and there be at least an element

$[y, x] = neut_{*_1}([y, x]) \in V$ such that $[x, x] = neut_{*_1}([y, x])$ for each x, y $\in$ V. Then $neut_{*_1}([y, x*_1y]) = neut_{*_1}([y, x])$ if and only if $[x, y] = anti_{*_1}[y, x]$

Proof:

$\Rightarrow$:We assume that there exists at least an element $[y, x] = neut_{*_1}([y, x]) \in V$ such that

$[x, x] = neut_{*_1}([y, x])$ for each x, y $\in$ V and $neut_{*_1}([y, x*_1y]) = neut_{*_1}([y, x])$. Thus,

$neut_{*_1}([y, x*_1y]) = [x*_1y, x*_1y]$. Also,

$neut_{*_1}([y, x*_1y]) = [x, x] *_1 [x, y] *_1 [y, x] *_1 [y, y]$, since $[.,.]$ is a bilinear mapping. Therefore,

$neut_{*_1}([y, x*_1y]) = [x, y] *_1 [y, x]$, since $[x, x] = neut_{*_1}([y, x])$ and $[y, y] = neut_{*_1}([x, y]$. Also,

$neut_{*_1}([y, x]) = [x, y] *_1 [y, x]$, since $neut_{*_1}([y, x*_1y]) = neut_{*_1}([y, x])$.

Thus, from Definition 2.2, $[x, y] = anti_{*_1}[y, x]$.

$\Leftarrow$:We assume that there exists at least an element $[y, x] = neut_{*_1}([y, x]) \in V$ such that

$[x, x] = neut_{*_1}([y, x])$ for each x, y $\in$ V and $[x, y] = anti_{*_1}[y, x]$. Then,

$neut_{*_1}([y, x]) = [x, y] *_1 [y, x]$, since $[x, y] = anti_{*_1}[y, x]$. Also,

$neut_{*_1}([y, x]) = [x, x] *_1 [x, y] *_1 [y, x] *_1 [y, y],$

since $[x, x] = neut_{*_1}([y, x])$ and $[y, y] = neut_{*_1}([x, y]$. Furthermore,

$neut_{*_1}([y, x]) = [x, x] *_1 [x, y] *_1 [y, x] *_1 [y, y] = [x *_1 y, x *_1 y],$ since $[.,.]$ is a bilinear mapping. Thus,

$neut_{*_1}([y, x]) = neut_{*_1}([y, x *_1 y]),$ since $[x, x] = neut_{*_1}([y, x]).$

Definition 3.6: Let $(V_1, *_1, \#_1)$ be a NTVS on NTF $(F_1, *_2, \#_2)$, the mapping $[.,.]_1: V_1 x V_1 \rightarrow V_1$ be a NT lie algebra on $(V_1, *_1, \#_1)$ and $(V_2, *_3, \#_3)$ be a NTVS on NTF $(F_2, *_4, \#_4)$, the mapping $[.,.]_2: V_2 x V_2 \rightarrow V_2$ be a NT lie algebra on $(V_2, *_3, \#_3)$. The mapping $\sigma: V_1 \rightarrow V_2$ is called a NT lie algebra homomorphism such that

$\sigma([a, b]_1) = [\sigma(a), \sigma(b)]_2$

Example 3.7: In Example 3.2, for V = {Ø, {a}, {b}, {a, b}}. (V, ∪, ∩) is a NTF and NTVS such that

$\quad neut_\cup(X) = X,$

$\quad anti_\cup(X) = Y, \ Y \subset X$ \hfill (1)

and

$\quad neut_\cap(X) = X,$

$\quad anti_\cap(X) = Y, \quad X \subset Y$ \hfill (2)

Also,

the mapping $[.,.]_1: V x V \rightarrow V, [A, B]_1 = A \cup B$ is a NT lie algebra on (V, ∪, ∩).

Now show that $[.,.]_2: V x V \rightarrow V, [A, B]_2 = A \cap B$ is a NT lie algebra on (V, ∪, ∩).

 i) a)

 $[A \cup (B \cap C), D]_2 = (A \cup (B \cap C)) \cap D = (A \cap D) \cup (A \cap B \cap D) = [A, D]_2 \cup (B \cap [C, D]_2).$

b) It can be show similarly to a).

ii) $[A, A]_2 = A \cap A = A = neut_A(A)$. So, there exists an element $A = neut_\cup(A) \in V$ such that $[A, A]_2 = neut_\cup(A)$, for each $A \in V$.

iii) $[A, [B, C]_2]_2 \cup [B, [C, A]_2]_2 \cup [C, [A, B]_2]_2 = A \cap B \cap C = neut_\cup(A \cap B \cap C)$. So, there exists an element $A \cap B \cap C = neut_\cup(A \cap B \cap C) \in V$ such that

$[A, [B, C]_2]_2 \cup [B, [C, A]_2]_2 \cup [C, [A, B]_2]_2 = neut_\cup(A \cap B \cap C)$, for each A, B, C ∈ V triplet. Similarly, there exists an element $A \cap B \cap C = neut_\cup(A \cap B \cap C) \in V$ such that

$[A, B], C]_2 \cup [B, C]_2, A]_2 \cup [C, A]_2, B]_2 = neut_\cup(A \cap B \cap C)$, for each A, B, C ∈ V triplet.

Therefore, $[A, B]_2 = A \cap B$ is a NT lie algebra on (V, ∪, ∩).

Also, let $\sigma: V \to V$, $\varphi(A) = A'$ (A' is complement of A) be a mapping. Now we show that

$\sigma([A, B]_1) = [\sigma(A), \sigma(B)]_2$.

$\sigma([A, B]_1) = \sigma(A \cup B) = (A \cup B)' = A' \cap B' = [A, B]_2 = [\sigma(A), \sigma(B)]_2$. Thus, $\sigma(A) = A'$ is a NT lie algebra homomorphism.

Definition 3.8: Let $(V, *_1, \#_1)$ be a NTVS on NTF $(F, *_2, \#_2)$, the mapping $[.,.]:VxV\to V$ be a NT lie algebra on $(V, *_1, \#_1)$ and $(S, *_1, \#_1)$ be a subvector space of $(V, *_1, \#_1)$. If for ∀ x, y ∈ S, [x, y] ∈ S, then $(S, *_1, \#_1)$ is called NT lie subalgebra of $(V, *_1, \#_1)$.

Example 3.9: In Example 3.2, for V = {∅, {a}, {b}, {a, b}}. (V, ∪, ∩) is a NTF and NTVS such that

$\quad neut_\cup(X) = X,$

$\quad anti_\cup(X) = Y, Y \subset X$ (1)

and

$\quad neut_\cap(X) = X,$

$\quad anti_\cap(X) = Y, \quad X \subset Y$ (2)

Also,

the mapping $[.,.]_1:VxV\to V$, $[A, B]_1 = A \cup B$ is a NT lie algebra on (V, ∪, ∩).

We take S = {∅, {a}}⊂ V. İt is clear that (S, ∪, ∩) is a subvector space of (V, ∪, ∩). Also,

for ∀ A, B ∈ S, $[A, B]_1 = A \cup B \in S$. Thus, (S, ∪, ∩) is a NT lie subalgebra of (V, ∪, ∩).

Definition 3.10: Let $(V, *_1, \#_1)$ be a NTVS on NTF $(F, *_2, \#_2)$, the mapping $[.,.]:VxV\to V$ be a NT lie algebra on $(V, *_1, \#_1)$ and $(S, *_1, \#_1)$ be a subvector space of $(V, *_1, \#_1)$. If for ∀ x ∈ V and y ∈ S, [x, y] ∈ S, then $(S, *_1, \#_1)$ is called NT lie ideal of $(V, *_1, \#_1)$.

Example 3.11: In Example 3.7, for V = {∅, {a}, {b}, {a, b}}. (V, ∪, ∩) is a NTF and NTVS such that

$neut_\cup(X) = X,$

$anti_\cup(X) = Y, \ Y \subset X$ (1)

and

$neut_\cap(X) = X,$

$anti_\cap(X) = Y, \ \ X \subset Y$ (2)

Also,

the mapping $[.,.]_2 : V \times V \rightarrow V$, $[A, B]_2 = A \cap B$ is a NT lie algebra on $(V, \cup, \cap)$.

We take $S = \{\emptyset, \{a\}\} \subset V$. İt is clear that $(S, \cup, \cap)$ is a subvector space of $(V, \cup, \cap)$. Also,

for $\forall \ A \in S$ and $B \in V$, $[A, B]_1 = A \cap B \in S$. Thus, $(S, \cup, \cap)$ is a NT lie ideal of $(V, \cup, \cap)$.

Definition 3.12: Let $(V, *_1, \#_1)$ be a NTVS on NTF $(F, *_2, \#_2)$, the mapping $[.,.] : V \times V \rightarrow V$ be a NT lie algebra on $(V, *_1, \#_1)$ and $(S, *_1, \#_1)$ be a NT lie ideal of $(V, *_1, \#_1)$. Then V/S $= \{x *_1 S : x \in V\}$ is called NT lie quotient algebra of $(V, *_1, \#_1)$. Also, $x *_1 S$ is called NT lie coset

and

$(x *_1 S) *_1 (y *_1 S) = (x *_1 y) *_1 S, \ x, y \in V;$

$\alpha \#_1 (x *_1 S) = (\alpha \#_1 x) *_1 S;$

$[x *_1 S, y *_1 S] = [x, y] *_1 S.$

Example 3.13: In Example 3.11,

for $V = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $S = \{\emptyset, \{a\}\}$, $(S, \cup, \cap)$ is a NT lie ideal of $(V, \cup, \cap)$. Also,

$V/S = \{A \cup S : A \in V\} = \{\emptyset \cup S, \{a\} \cup S, \{b\} \cup S, \{a, b\} \cup S\}.$

Theorem 3.14: Let $(V, *_1, \#_1)$ be a NTVS on NTF $(F, *_2, \#_2)$, the mapping $[.,.] : V \times V \rightarrow V$ be a NT lie algebra on $(V, *_1, \#_1)$ and V/S $= \{x *_1 S : x \in V\}$ be a NT lie quotient algebra of $(V, *_1, \#_1)$. Then, $[.,.]$ is a NT lie algebra on V/S $= \{x *_1 S : x \in V\}$ with

$(x *_1 S) *_1 (y *_1 S) = (x *_1 y) *_1 S, \ x, y \in V;$

$\alpha \#_1 (x *_1 S) = (\alpha \#_1 x) *_1 S, \ \alpha \in F$ and $x \in V;$

$[x *_1 S, y *_1 S] = [x, y] *_1 S, \ x, y \in V.$

Proof: It is clear that $(V/S, *_1, \#_1)$ is a NTVS on NTF $(F, *_2, \#_2)$ with

$(x*_1 S) *_1 (y*_1 S) = (x*_1 y)*_1 S$, $x, y \in V$; and $\alpha \#_1 (x*_1 S) = (\alpha \#_1 x)*_1 S$, $\alpha \in F$; $x \in V$. Also,

i)

a) $[(x*_1 S) *_1 (\alpha \#_1 (y*_1 S))), z*_1 S] =$

   $[\alpha \#_1 (x *_1 y*_1) *_1 S, z*_1 S] =$

   $[\alpha \#_1 (x *_1 y*_1), z] *_1 S =$

   $([x, z] *_1 (\alpha \#_1 [y, z])) *_1 S =$

   $([x, z] *_1 (\alpha \#_1 [y, z])) *_1 S =$

   $([x, z] *_1 S) *_1 (\alpha \#_1 [y, z]) *_1 S) =$

   $([x*_1 S, z*_1 S]) *_1 (\alpha \#_1 [y*_1 S, z*_1 S])$.

b) It can be show similarly to a).

ii) $[x*_1 S, x*_1 S] = [x, x] *_1 S = neut_{*_1}(t) *_1 S$ since $[.,.]$ is a NT lie algebra on $(V, *_1, \#_1)$. Thus, there exists at least an element $t*_1 S = neut_{*_1}(t) *_1 S \in V/S$ such that $[x*_1 S, x*_1 S] = [x, x] *_1 S = neut_{*_1}(t) *_1 S$, for each $x*_1 S \in V/S$.

iii) $[x*_1 S, [y*_1 S, z*_1 S]] *_1 [y*_1 S, [z*_1 S, x*_1 S]] *_1 [z*_1 S, [x*_1 S, y*_1 S]] =$

   $[x*_1 S, [y, z] *_1 S] *_1 [y*_1 S, [z, x] *_1 S] *_1 [z*_1 S, [x, y] *_1 S] =$

   $([x, [y, z]] *_1 S) *_1 ([y, [z, x]] *_1 S) *_1 ([z, [x, y]] *_1 S) =$

   $([y, z]] *_1 [y, [z, x]] *_1 [z, [x, y]]) *_1 S = neut_{*_1}(t) *_1 S$.

Thus, there exists at least an element $t *_1 S = neut_{*_1}(t) *_1 S \in V/S$ such that $[x*_1 S, [y*_1 S, z*_1 S]] *_1 [y*_1 S, [z*_1 S, x*_1 S]] *_1 [z*_1 S, [x*_1 S, y*_1 S]] = neut_{*_1}(t) *_1 S$, for each $x*_1 S, y*_1 S, z*_1 S \in V/S$.

Similarly, there exists at least an element $t *_1 S = neut_{*_1}(t) *_1 S \in V/S$ such that $[x*_1 S, y*_1 S], z*_1 S]] *_1 [y*_1 S, z*_1 S], x*_1 S]] *_1 [z*_1 S, x*_1 S], y*_1 S]] = neut_{*_1}(t) *_1 S$ for each $x, y, z \in V$. Therefore, $[.,.]$ is a NT lie algebra on $V/S = \{x*_1 S: x \in V\}$.

4. Conclusions

In this paper, we obtained NT lie algebra. Also, we defined NT lie algebra homomorphism, NT lie subalgebra, NT lie coset and NT lie ideal. Thus, we have added a new structure to NT structure and we gave rise to a new field or research called NT lie algebra. Also, thanks to NT lie algebras and their properties, researchers can obtain isomorphism theorems for NT lie algebras, NT lie groups, representation of NT lie algebras, NT simple lie algebras, NT free lie algebra.

Abbreviation

NT: Neutrosophic triplet

NTS: Neutrosophic triplet set

NTG: Neutrosophic triplet group

NTF: Neutrosophic triplet field

NTVS: Neutrosophic triplet vector space

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Chapter Seven
Neutrosophic Triplet b - Metric Space

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Abstract

In this chapter, we firstly obtain neutrosophic triplet b- metric space. Also, we give some definitions and examples for neutrosophic triplet b - metric space. Furthermore, we obtain some properties and we prove these properties. Also, we show that neutrosophic triplet b - metric space is different from classical b - metric space and neutrosophic triplet metric space.

Keywords: neutrosophic triplet sets, neutrosophic triplet metric spaces, b - metric space, neutrosophic triplet b - metric spaces

1. Introduction

Smarandache introduced neutrosophy in 1980, which studies a lot of scientific fields. In neutrosophy, there are neutrosophic logic, set and probability in [1]. Neutrosophic logic is a generalization of a lot of logics such as fuzzy logic in [2] and intuitionistic fuzzy logic in [3]. Neutrosophic set is showed by (t, i, f) such that "t" is degree of membership, "i" is degree of indeterminacy and "f" is degree of non-membership. Also, a lot of researchers have studied neutrosophic sets in [4-9]. Furthermore, Smarandache and Ali obtained neutrosophic triplet (NT) in [10] and they introduced NT groups in [11]. For every element "x" in neutrosophic triplet set A, there exist a neutral of "a" and an opposite of "a". Also, neutral of "x" must different from the classical unitary element. Therefore, the NT set is different from the classical set. Furthermore, a NT "x" is showed by <x, neut(x), anti(x)>. Also, many researchers have introduced NT structures in [12-20].

Bakhtin obtained b - metric spaces in [21]. The b - metric is generalized of classical metric space. b - metric space is used mostly for fixed point theory. Also, researchers studied partial metric space in [22-27]. Recently, Alqahtani *et al.* studied Fisher – type fixed point results in b – metric spaces in [28] and Oawaqneh *et al.* obtained fixed point theorems for (a, k, θ) – contractive multi – valued mapping in b – metric space and applications.

In this chapter, we obtain NT b - metric space. In section 2; we give definitions of b - metric space in [21], NT set in [11], NT metric space in [13]. In section 3, we introduce

NT b - metric space and we give some properties and examples for NT b - metric space. Also, we show that NT b - metric space is different from the classical b - metric space and NT metric space. Furthermore, we show relationship between NT metric spaces and NT partial metric spaces with NT b - metric space. Also, we give definition of convergent sequence, Cauchy sequence and complete space for NT partial v-generalized metric space. In section 4, we give conclusions.

2. Basic and Fundamental Concepts

Definition 2.1: [21] Let N be a nonempty set and d_b:NxN$\rightarrow \mathbb{R}$ be a function. If d is satisfied the following properties, then (N, d_b) is called a b - metric space. For n, m, $c_1, c_2, \dots , c_v \in$ N,

i) d_b(n, m) $\geq$ 0 and d_b(n, m) = 0 $\Leftrightarrow$ n = m;

ii) d_b(n, m)= d_b(m, n);

iii) d_b(n, m) $\leq$ k.[d_b(n,k)+ d_b(k, m)], k $\in \mathbb{R}^+$ such that k $\geq$ 1

Definition 2.2: [11] Let # be a binary operation. (X, #) is a NT set (NTS) such

i) There must be neutral of "x" such x#neut(x) = neut(x)#x = x, x $\in$ X.

ii) There must be anti of "x" such x#anti(x) = anti(x)#x = neut(x), x $\in$ X.

Furthermore, a NT "x" is showed with (x, neut(x), anti(x)).

Also, neut(x) must different from classical unitary element.

Definition 2.3: [13] A NT metric on a NTS (N, *) is a function d:NxN$\rightarrow \mathbb{R}$ such for every n, m, s $\in$ N,

i) n * m $\in$ N

ii) d(n, m) $\geq$ 0

iii) If n = m, then d(n, m) = 0

iv) d(n, m) = d(n, m)

v) If there exists at least an element s $\in$ N for n, m $\in$ N pair such that

d(n, m) $\leq$ d(n, m*neut(s)), then d(n, m*neut(s)) $\leq$ d(n, s)+ d_T(s, n).

Definition 2.4: [17] Let (A, #) be a NTS and m#n $\in$ A, $\forall$ m, n $\in$ A. NT partial metric (NTPM) is a map p_N: A x A $\rightarrow \mathbb{R}^+\cup\{0\}$ such that $\forall$ m, n, k $\in$ A

i) p_N(m, n) $\geq p_N$(n, n)$\geq$0

ii) If p_N(m, m) $= p_N$(m, n) $= p_N$(n, n) = 0, then there exits any m, n ∈ A pair such that m = n.

iii) p_N(m, n) = p_N(n, m)

iv) If there exists at least an element n∈A for each m, n ∈A pair such that

p_N(m, k) $\leq p_N$(m, k#neut(n)), then p_N(m, k#neut(n)) $\leq p_N$(m, n) + p_N(n, k) - p_N(n, n).

Also, ((A, #), p_N) is called NTPM space (NTPMS).

3. Neutrosophic Triplet b - Metric Space

Definition 3.1: Let (N, *) be a NTS. A NT b – metric (NTBM) on N is a function such that d_b:NxN→ $\mathbb{R}$ such every n, m, t ∈ N;

i) n*m ∈ N,

ii) If n = m, then d_b(m, n) = 0 and d_b(n, m) $\geq$ 0.

iii) d_b(n, m) = d_b(m, n),

iv) If there exists at least an element s ∈ N for each n, m ∈ N pair such that

d_b(n, m) $\leq d_b$(n, m*neut(s)), then

d_b(n, m*neut(s)) $\leq$ k.[d_b(n, s) + d_b(s, n)]. Where, k $\geq$ 1 and k ∈ $\mathbb{R}$.

Furthermore, ((N, *), d_b) is called NTBM space (NTBMS).

Also, if we take k = 2, then NTBMS is showed that NT2MS.

Example 3.2: Let N = {0, 2, 3, 4} be a set. (N, .) is a NTS under multiplication module 12 in ($\mathbb{Z}_6$, .). Also, NT are (0, 0, 0), (2, 2, 2), (3, 3, 3), (4, 4, 4) and (2, 4, 2).

Then we take that d_b: NxN→N is a function such that d_b(k, m)=$\left| 2^k - 2^m \right|$.

Now we show that d_b is a NTBM.

i) It is clear that k.m ∈ N, for every k, m ∈ N.

ii) If k = m, then d_b(k, m)=$\left| 2^k - 2^m \right|$ = $\left| 2^k - 2^k \right|$ = 0. Also, d_b(k, m)=$\left| 2^k - 2^m \right|$ $\geq$ 0.

iii) d_b(k, m) =$\left| 2^k - 2^m \right|$ =$\left| 2^m - 2^k \right|$ = d_b(m, k).

iv)

* It is clear that $d_b(0, 0) \le d_b(0, 0.0) = d_b(0, 0)$. Also, $d_b(0, 0) = 0$. Thus,

$d_b(0, 0) \le 2.[d_b(0, 0) + d_b(0, 0)]$.

* It is clear that $d_b(0, 3) \le d_b(0, 3.3) = d_b(0, 3)$. Also, $d_b(0, 3) = 7$ and $d_b(3, 3) = 0$. Thus,

$d_b(0, 3) \le 2.[d_b(0, 3) + d_b(3, 3)]$.

* It is clear that $d_b(0, 2) \le d_b(0, 2.4) = d_b(0, 2)$. Also, $d_b(0, 2) = 3$, $d_b(0, 4) = 15$ and $d_b(2, 4) = 12$. Thus,

$d_b(0, 2) \le 2.[d_b(0, 4) + d_b(4, 2)]$.

* It is clear that $d_b(0, 4) \le d_b(0, 4.4) = d_b(0, 4)$. Also, $d_b(0, 4) = 15$ and $d_b(4, 4) = 0$. Thus,

$d_b(0, 4) \le 2.[d_b(0, 4) + d_b(4, 4)]$.

* It is clear that $d_b(3, 3) \le d_b(3, 3.2) = d_b(3, 0)$. Also, $d_b(3, 0) = 7$, $d_b(3, 2) = 4$ and $d_b(3, 3) = 0$. Thus,

$d_b(3, 3) \le 2.[d_b(3, 2) + d_b(2, 3)]$.

* It is clear that $d_b(2, 2) \le d_b(2, 2.3) = d_b(2, 0)$. Also, $d_b(3, 2) = 4$ and $d_b(2, 2) = 0$. Thus,

$d_b(2, 2) \le 2.[d_b(2, 3) + d_b(3, 2)]$.

* It is clear that $d_b(4, 4) \le d_b(4, 4.2) = d_b(4, 2)$. Also, $d_b(4, 2) = 12$ and $d_b(4, 4) = 0$. Thus,

$d_b(4, 4) \le 2.[d_b(4, 2) + d_b(2, 4)]$.

* It is clear that $d_b(3, 2) \le d_b(3, 2.3) = d_b(3, 0)$. Also, $d_b(3, 0) = 7$, $d_b(3, 2) = 4$ and $d_b(3, 3) = 0$. Thus,

$d_b(3, 2) \le 2.[d_b(3, 3) + d_b(3, 2)]$.

* It is clear that $d_b(3, 4) \le d_b(3, 4.4) = d_b(3, 0)$. Also, $d_b(3, 0) = 7$, $d_b(3, 4) = 8$ and $d_b(4, 4) = 0$. Thus,

$d_b(3, 4) \le 2.[d_b(3, 4) + d_b(4, 4)]$.

* It is clear that $d_b(4, 2) \le d_b(4, 2.3) = d_b(4, 0)$. Also, $d_b(4, 0) = 15$, $d_b(3, 4) = 8$, $d_b(3, 2) = 4$ and $d_b(4, 2) = 12$. Thus,

$d_b(4, 2) \le 2.[d_b(4, 3) + d_b(3, 2)]$.

Therefore, $((N, .), d_b)$ is A NT2MS.

Corollary 3.3: NTBMS is different from the classical metric space since for triangle inequality and * binary operation.

Corollary 3.4: NTBMS is different from NTMS since for triangle inequality.

Corollary 3.5: In Definition 3.1, if we take k = 1, then each NTBMS is a NTMS.

Corollary 3.6: From Corollary 3.5, we can define a NTBMS with each NTMS.

Corollary 3.7: From Corollary 3.6, each NTMS satisfies all properties of NTBMS.

Example 3.8: In Example 3.2, $((N, .), d_b)$ is a NT2MS. Now we show that $((N, .), d_b)$ is a NTMS.

It is clear that $((N, .), d_b)$ satisfies conditions i, ii, iii, iv in Definition 2.3.

v)

 * It is clear that $d_b(0, 0) \leq d_b(0, 0.0) = d_b(0, 0)$. Also, $d_b(0, 0) = 0$. Thus,

$d_b(0, 0) \leq d_b(0, 0) + d_b(0, 0)$.

* It is clear that $d_b(0, 3) \leq d_b(0, 3.3) = d_b(0, 3)$. Also, $d_b(0, 3) = 7$ and $d_b(3, 3) = 0$. Thus,

$d_b(0, 3) \leq d_b(0, 3) + d_b(3, 3)$.

* It is clear that $d_b(0, 2) \leq d_b(0, 2.4) = d_b(0, 2)$. Also, $d_b(0, 2) = 3$, $d_b(0, 4) = 15$ and $d_b(2, 4) = 12$. Thus,

$d_b(0, 2) \leq d_b(0, 4) + d_b(4, 2)$.

* It is clear that $d_b(0, 4) \leq d_b(0, 4.4) = d_b(0, 4)$. Also, $d_b(0, 4) = 15$ and $d_b(4, 4) = 0$. Thus,

$d_b(0, 4) \leq d_b(0, 4) + d_b(4, 4)$.

* It is clear that $d_b(3, 3) \leq d_b(3, 3.2) = d_b(3, 0)$. Also, $d_b(3, 0) = 7$, $d_b(3, 2) = 4$ and $d_b(3, 3) = 0$. Thus,

$d_b(3, 3) \leq d_b(3, 2) + d_b(2, 3)$.

* It is clear that $d_b(2, 2) \leq d_b(2, 2.3) = d_b(2, 0)$. Also, $d_b(3, 2) = 4$ and $d_b(2, 2) = 0$. Thus,

$d_b(2, 2) \leq d_b(2, 3) + d_b(3, 2)$.

* It is clear that $d_b(4, 4) \leq d_b(4, 4.2) = d_b(4, 2)$. Also, $d_b(4, 2) = 12$ and $d_b(4, 4) = 0$. Thus,

$d_b(4, 4) \leq d_b(4, 2) + d_b(2, 4)$.

* It is clear that $d_b(3, 2) \leq d_b(3, 2.3) = d_b(3, 0)$. Also, $d_b(3, 0) = 7$, $d_b(3, 2) = 4$ and $d_b(3, 3) = 0$. Thus,

$d_b(3, 2) \leq d_b(3, 3) + d_b(3, 2)$.

* It is clear that $d_b(3, 4) \leq d_b(3, 4.4) = d_b(3, 0)$. Also, $d_b(3, 0) = 7$, $d_b(3, 4) = 8$ and $d_b(4, 4) = 0$. Thus,

$d_b(3, 4) \leq d_b(3, 4) + d_b(4, 4)$.

* It is clear that $d_b(4, 2) \leq d_b(4, 2.3) = d_b(4, 0)$. Also, $d_b(4, 0) = 15$, $d_b(3, 4) = 8$, $d_b(3, 2) = 4$ and $d_b(4, 2) = 12$. Thus,

$d_b(4, 2) \leq d_b(4, 3) + d_b(3, 2)$.

Therefore, $((N, .), d_b)$ is A NTMS.

Theorem 3.9: Let $((N, \#), d_b)$ be a NTBMS. If the following condition is satisfied, then $((N, \#), d_b)$ is a NTPMS.

 If $d_b(m, n) = 0$, then

there exits any m, n $\in$ A pair such that m = n.   (1)

Proof: We show that $((N, \#), d_b)$ satisfies conditions of NTPMS. From Definition 2.4,

i) From Definition 3.1, it is clear that n#m $\in$ N, for every n, m $\in$ N.

ii) It is clear that $d_b(m, n) \geq d_b(n, n) \geq 0$. Because, from Definition 3.1, $d_b(n, n) = 0$.

iii) From Definition 3.1, $d_b(n, n) = d_b(m, m) = 0$. Also, from (1), if $d_b(m, n) = 0$, then m = n. Thus, Thus, if $d_b(m, m) = d_b(m, n) = d_b(n, n) = 0$, then there exits any m, n $\in$ A pair such that m = n.

iv) From Definition 3.1, $d_b(m, n) = d_b(n, m)$

v) From Definition 3.1, if there exists at least an element s $\in$ N for each n, m $\in$ N pair such that $d_b(n, m) \leq d_b(n, m*neut(s))$, then $d_b(n, m*neut(s)) \leq k.[d_b(n, s) + d_b(s, n)]$. Where, k $\geq$ 1 and k $\in$ $\mathbb{R}$. Thus, we can take $d_b(n, m*neut(s)) \leq d_b(n, s) + d_b(s, n)$. Because k $\geq$ 1 and k $\in$ $\mathbb{R}$. Also, we can take

d_b(n, m*neut(s)) $\leq d_b$(n, s) + d_b(s, n) - d_b(n, n) since d_b(n, n) = 0.

Therefore, ((N, #), d_b) is a NTPMS.

Theorem 3.10: Let ((N, #), d_b) be a NTBMS. Then, d(a, b) = $\frac{d_b(a,b)}{d_b(a,b)+1}$ is a NTBMS.

Proof:

i) It is clear that a#b $\in$ N.

ii) If a = b, then d_b(a, b) = 0. Because ((N, #), d_b) be a NTBMS. Thus,

if a =b, d(a, b) = $\frac{d_b(a,b)}{d_b(a,b)+1}$ = $\frac{0}{0+1}$ = 0. Also, d(a, b) = $\frac{d_b(a,b)}{d_b(a,b)+1}$ $\geq$ 0.

iii) d_b(a, b) = d_b(b, a), since ((N, #), d_b) be a NTBMS. Thus,

d(a, b) = $\frac{d_b(a,b)}{d_b(a,b)+1}$ = $\frac{d_b(b,a)}{d_b(b,a)+1}$ = d(b, a).

iv) If there exists at least an element c $\in$ N for each a, b $\in$ N pair such that d_b(a, b) $\leq d_b$(a, b*neut(c)), then

d_b(a, b*neut(c)) $\leq$

k.[d_b(a, c) + d_b(c, b)], since ((N, #), d_b) be a NTBMS. Where, k $\geq$ 1 and k $\in \mathbb{R}$.   (2)

Thus, if there exists at least an element c $\in$ N for a, b $\in$ N pair such that d_b(a, b) $\leq d_b$(a, b*neut(c)), then

d(a, b) = $\frac{d_b(a,b)}{d_b(a,b)+1}$ $\leq$ $\frac{k.[d_b(a,c) + d_b(c,b)]}{k.[d_b(a,c) + d_b(c,b)]+1}$ = $\frac{k.d_b(a,c)}{k.[d_b(a,c) + d_b(c,b)]+1}$ + $\frac{k.d_b(c,b)}{k.[d_b(a,c) + d_b(c,b)]+1}$

$\leq$ $\frac{k.d_b(a,c)}{d_b(a,c)+1}$ + $\frac{k.d_b(c,b)}{d_b(c,b)+1}$ = k.[d(a, c)+d(c, b)]. Because, k $\geq$ 1, k $\in \mathbb{R}$ and (2).

Thus, d(a, b) = $\frac{d_b(a,b)}{d_b(a,b)+1}$ is a NTBMS.

Definition 3.11: Let ((N, #), d_b) be a NTBMS and $\{x_n\}$ be a sequence in NTBMS and m $\in$ N. If there exist k $\in \mathbb{N}$ for every ε>0 such that

d_b(m, $\{x_n\}$) < ε

then $\{x_n\}$ converges to m. where n $\geq$ k. Also, it is showed that

$\lim\limits_{n\to\infty} x_n$ = m or $x_n \to$ m.

Definition 3.12: Let $((N, \#), d_b)$ be a NTBMS and $\{x_n\}$ be a sequence in NTBMS. If there exist a k ∈ ℕ for every ε>0 such that

$d_b(\{x_m\}, \{x_n\})< ε$

then $\{x_n\}$ is a Cauchy sequence in NTBMS. Where, n ≥m≥k.

Definition 3.13: Let $((N, \#), d_b)$ be a NTBMS and $\{x_n\}$ be a Cauchy sequence in NTBMS. NTBMS is called complete ⇔ every $\{x_n\}$ converges in NTBMS.

Conclusions

In this chapter, we obtained NTBMS. We also show that NTBMS is different from the NTMS and classical b - metrics. Also, we gave some properties for NTBMS. Thus, we have added a new structure to NT structure and we gave rise to a new field or research called NTBMS. Also, thanks to NTBMS researcher can obtain new structure and properties. For example, NT partial b – metric space, NT v – generalized b – metric, NT partial v – generalized b – metric NT b - normed space, NT b inner product space and NT fixed point theorems for NTBMS.

Abbreviations

NT: Neutrosophic triplet

NTS: Neutrosophic triplet set

NTM: Neutrosophic triplet metric

NTMS: Neutrosophic triplet metric space

NTPM: Neutrosophic triplet partial metric

NTPMS: Neutrosophic triplet partial metric space

NTBM: Neutrosophic triplet b – metric

NTBMS: Neutrosophic triplet b – metric space

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SECTION TWO

Decision Making

Chapter Eight

Multiple Criteria Decision Making in Architecture Based on Q-Neutrosophic Soft Expert Multiset

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Abstract: We will extend this further by presenting a novel concept of Q-neutrosophic soft expert multiset, and define the associated related concepts and basic operations of subset, intersection, union, complement, OR and AND along with illustrative examples, and study some related properties with supporting proofs. Then, we construct an algorithm based on this concept. We illustrate the feasibility of the new method by an example in architecture. Finally, a comparison of the proposed method to existing methods is furnished to verify the effectiveness of our novel concept.

Keywords: Decision making; Neutrosophic soft expert sets; Neutrosophic soft expert multiset; Q-fuzzy set, architecture.

1. Introduction

In general evaluation, architectural practice is an area where personal opinion is intense, and the judgments reached as a result of the perceptions and reactions that occur during design are shaped according to individual emotions and thoughts. In this respect, it is very important that the model to help the decision making process be effective in expressing the human dimension and uncertainties in the evaluation of the designs obtained after the architectural design process. Intuitionistic fuzzy sets were introduced by Atanassov [1], followed by Molodtsov [2] on soft set and neutrosophy logic [3] and neutrosophic sets [4] by Smarandache. The term neutrosophy means knowledge of neutral thought and this neutral represents the main distinction between fuzzy and intuitionistic fuzzy logic and set. Presently, work on soft set theory is progressing rapidly. Various operations and applications of soft sets were developed rapidly including multi-adjoint t-concept lattices[5], signatures: definitions, operators and applications to fuzzy modelling [6], fuzzy inference system optimized by genetic algorithm for robust face and pose detection [7], fuzzy multi-objective modeling of effectiveness and user experience in online advertising [8], possibility fuzzy soft set [9], soft multiset theory [10], multiparameterized soft set [11], soft intuitionistic fuzzy sets [12], Q-fuzzy soft sets [13–15], and multi Q-fuzzy sets [16–18], thereby opening avenues to many applications [19, 20]. Later, Maji [21] introduced a more generalized concept, which is a combination of neutrosophic sets and soft sets and studied its properties. Alkhazaleh and Salleh [22]

defined the concept of fuzzy soft expert set, which were later extended to vague soft expert set theory [23], generalized vague soft expert set [24] and multi Q-fuzzy soft expert set [25]. Şahin et al. [26] introduced neutrosophic soft expert sets, while Hassan et al. [27] extended it further to Q-neutrosophic soft expert set, Broumi et al. [28] defined neutrosophic parametrized soft set theory and its decision making, Deli [29] introduced refined neutrosophic sets and refined neutrosophic soft sets.

Since membership values are inadequate for providing complete informationin some real problems which has different membership values for each element, different generalization of fuzzy sets, intuitionistic fuzzy sets and neutrosophic sets have been introduced is called multi fuzzy set [30], intuitionistic fuzzy multiset [31] and neutrosophic multiset [32,33], respectively. In the multisets an element of a universe can be constructed more than once with possibly the same or different membership values. Some work on the multi fuzzy set [34,35], on intuitionistic fuzzy multiset [36-39] and on neutrosophic multiset [40-43] have been studied. The above set theories have been applied to many different areas including real decision making problems [44-52]. The aim of this paper, besides the objective evaluation, a decision making model that can be effective in expressing the subjective evaluations within the structure of architecture (mass, spatial, semantic, form and experience) has been developed.

Finally, we apply this new concept to solve a decision-making problem in architecture and compare it with other existing methods.

2. Preliminaries

In this section we review the basic definitions of a neutrosophic set, neutrosophic soft expert multiset, neutrosophic soft expert sets, Q-neutrosophic soft expert sets required as preliminaries.

Definition 2.1 ([4]) A neutrosophic set A on the universe of discourse $\mathcal{U}$ is defined as $A = \{< u, (\mu_A(u), v_A(u), w_A(u)) > : u \in U, \mu_A(u), v_A(u), w_A(u) \in [0,1]\}$. There is no restriction on the sum of $\mu_A(u)$; $v_A(u)$ and $w_A(u)$, so $0^- \leq \mu_\mathcal{A}(u) + v_\mathcal{A}(u) + w_\mathcal{A}(u) \leq 3^+$.

Definition 2.2 ([21]) Let $\mathcal{U}$ be an initial universe set and E be a set of parameters. Consider $A \subseteq E$. Let $NS(\mathcal{U})$ denotes the set of all neutrosophic sets of $\mathcal{U}$. The collection (F, A) is termed to be the neutrosophic soft set over $\mathcal{U}$, where F is a mapping given by $F: A \rightarrow NS(\mathcal{U})$.

Definition 2.3 ([22]) $\mathcal{U}$ is an initial universe, E is a set of parameters X is a set of experts (agents), and $O = \{\text{agree} = 1, \text{disagree} = 0\}$ a set of opinions. Let $Z = E \times X \times O$ and $A \subseteq Z$. A pair (F, A) is called a soft expert set over $\mathcal{U}$, where F is mapping given by $F: A \rightarrow P(\mathcal{U})$ where $P(\mathcal{U})$ denote the power set of $\mathcal{U}$.

Definition 2.4 ([26]) A pair (F, A) is called a neutrosophic soft expert set over $\mathcal{U}$, where F is mapping given by

$$F: A \to P(\mathcal{U}) \tag{1}$$

where $P(\mathcal{U})$ denotes the power neutrosophic set of U.

Definition 2.5 ([26]) The complement of a neutrosophic soft expert set (F, A) denoted by $(F, A)^c$ and is defined as $(F, A)^c = (F^c, \neg A)$ where $F^c = \neg A \to P(\mathcal{U})$ is mapping given by $F^c(x) =$ neutrosophic soft expert complement with $\mu_{F^c(x)} = w_{F(x)}$, $v_{F^c(x)} = v_{F(x)}$, $w_{F^c(x)} = \mu_{F(x)}$.

Definition 2.6 ([26]) The agree-neutrosophic soft expert set $(F, A)_1$ over $\mathcal{U}$ is a neutrosophic soft expert subset of (F, A) is defined as

$$(F, A)_1 = \{F_1(m): m \in E \times X \times \{1\}\}. \tag{2}$$

Definition 2.7 ([26]) The disagree-neutrosophic soft expert set $(F, A)_0$ over $\mathcal{U}$ is a neutrosophic soft expert subset of (F, A) is defined as

$$(F, A)_0 = \{F_0(m): m \in E \times X \times \{0\}\}. \tag{3}$$

Definition 2.8 ([26]) Let (H, A) and (G, B) be two NSESs over the common universe U. Then the union of (H, A) and (G, B) is denoted by "$(H, A) \tilde{\cup} (G, B)$" and is defined by $(H, A) \tilde{\cup} (G, B) = (K, C)$, where $C = A \cup B$ and the truth-membership, indeterminacy-membership and falsity-membership of (K, C) are as follows:

$$\mu_{K(e)}(m) = \begin{cases} \mu_{H(e)}(m) & , \quad if\ e \in A - B, \\ \mu_{G(e)}(m) & , \quad if\ e \in B - A, \\ max\left(\mu_{H(e)}(m), \mu_{G(e)}(m)\right), & if\ e \in A \cap B. \end{cases}$$

$$v_{K(e)}(m) = \begin{cases} v_{H(e)}(m) & , \quad if\ e \in A - B, \\ v_{G(e)}(m) & , \quad if\ e \in B - A, \\ \dfrac{v_{H(e)}(m) + v_{G(e)}(m)}{2} & , \quad if\ e \in A \cap B. \end{cases} \tag{4}$$

$$w_{K(e)}(m) = \begin{cases} w_{H(e)}(m) & , \quad if\ e \in A - B, \\ w_{G(e)}(m) & , \quad if\ e \in B - A, \\ min\left(w_{H(e)}(m), w_{G(e)}(m)\right), & if\ e \in A \cap B. \end{cases}$$

Definition 2.9 ([26]) Let (H, A) and (G, B) be two NSESs over the common universe U. Then the intersection of (H, A) and (G, B) is denoted by "$(H, A) \tilde{\cap} (G, B)$" and is defined

$by (H, A) \tilde{\cap} (G, B) = (K, C)$, where $C = A \cap B$ and the truth-membership, indeterminacy-membership and falsity-membership of (K, C) are as follows:

$$\mu_{K(e)}(m) = \min\left(\mu_{H(e)}(m), \mu_{G(e)}(m)\right),$$

$$v_{K(e)}(m) = \frac{v_{H(e)}(m) + v_{G(e)}(m)}{2}, \qquad (5)$$

$$w_{K(e)}(m) = \max\left(w_{H(e)}(m), w_{G(e)}(m)\right),$$

$if\ e \in A \cap B.$

Definition 2.10 ([29]) Let $\mathcal{U}$ be a universe. A neutrosophic multiset set (Nms) A on $\mathcal{U}$ can be defined as follows :

$A = \{$

$\prec u, \left(\mu_A^1(u), \mu_A^2(u), \dots, \mu_A^p(u)\right), \left(v_A^1(u), v_A^2(u), \dots, v_A^p(u)\right), \left(w_A^1(u), w_A^2(u), \dots, w_A^p(u)\right)$

$\succ : u \in \mathcal{U}\}$

where,

$$\mu_A^1(u), \mu_A^2(u), \dots, \mu_A^p(u): \mathcal{U} \to [0,1],$$
$$v_A^1(u), v_A^2(u), \dots, v_A^p(u): \mathcal{U} \to [0,1],$$

and

$$w_A^1(u), w_A^2(u), \dots, w_A^p(u): \mathcal{U} \to [0,1],$$

such that

$$0 \leq \sup\mu_A^i(u) + \sup v_A^i(u) + \sup w_A^i(u) \leq 3$$

$(i = 1,2, \dots, P)$ and

$\left(\mu_A^1(u), \mu_A^2(u), \dots, \mu_A^p(u)\right), \left(v_A^1(u), v_A^2(u), \dots, v_A^p(u)\right) and \left(w_A^1(u), w_A^2(u), \dots, w_A^p(u)\right)$

Is the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of the element u, respectively. Also, P is called the dimension (cardinality) of Nms A, denoted $d(A)$. We arrange the truth- membership sequence in decreasing order but the corresponding indeterminacy- membership and falsity-membership sequence may not be in decreasing or increasing order.

The set of all Neutrosophic multisets on $\mathcal{U}$ is denoted by NMS($\mathcal{U}$).

Definition 2.11 ([25]) Let I be unit interval and k be a positive integer. A multi Q -fuzzy set $\tilde{A}_Q$ in V and a non-empty set Q is a set of ordered sequences $\tilde{A}_Q = \{(v. q), \mu_i(v, q): v \in V, q \in Q\}$ where

$$\mu_i: V \times Q \to I^k, \quad i = 1,2, \dots, k.$$

The function $\left(\mu_1(v, q), \mu_2(v, q), \dots, \mu_k(v, q)\right)$ is called the membership function of multi Q- fuzzy set $\tilde{A}_Q$: and $\mu_1(v, q) + \mu_2(v, q) + \cdots + \mu_k(v, q) \leq 1, k$ is called the dimension of $\tilde{A}_Q$. The set of all multi Q- fuzzy sets of dimension k in V and Q is denoted by $M^k QF(V)$.

3. Q-Neutrosophic Soft Expert Multiset Sets

We will now propose the definition of Q-neutrosophic soft expert multiset ($QNSEMS$) and propose some of its properties. Throughout this paper, $\mathbb{U}$ is an initial universe, E is a set of parameters, Q be a set of supply, X is a set of experts (agents), and $O = \{\text{agree} = 1, \text{disagree} = 0\}$ a set of opinions. Let $Z = E \times X \times O$ and $G \subseteq Z$.

Definition 3.1. (F_Q, G) is called a Q-neutrosophic soft expert multiset over $\mathbb{U}$, where F_Q is the mapping

$F_Q: G \to QNSEMS$ such that $QNSEMS$ is the set of all $QNSEMS$ over $\mathbb{U}$.

Example 3.2. Assume that a construction company making new moving structures wishes to receive feedback of a few experts. Let $\mathbb{U} = \{\mathbb{u}_1\}$ is a set of moving structure, $Q = \{q_1, q_2\}$ be the set of suppliers and $E = \{e_1 = tempature, e_2 = time\}$ is a set of decision parameters. Let $X = \{p, r\}$ be set of experts. Suppose that

$F_Q(e_1, p, 1)$
$$= \left\{ \left(\frac{(\mathbb{u}_1, q_1)}{(0.4,0.3,\ldots,0.2),(0.5,0.7,\ldots,0.2),(0.6,0.1,\ldots,0.3)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.7,0.2,\ldots,0.4),(0.4,0.6,\ldots,0.3),(0.2,0.3,\ldots,0.4)} \right) \right\}$$

$F_Q(e_1, r, 1)$
$$= \left\{ \left(\frac{(\mathbb{u}_1, q_1)}{(0.3,0.2,\ldots,0.5),(0.8,0.1,\ldots,0.4),(0.5,0.6,\ldots,0.2)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.8,0.5,\ldots,0.3),(0.5,0.7,\ldots,0.2),(0.3,0.1,\ldots,0.2)} \right) \right\}$$

$F_Q(e_2, p, 1)$
$$= \left\{ \left(\frac{(\mathbb{u}_1, q_1)}{(0.7,0.3,\ldots,0.6),(0.3,0.2,\ldots,0.6),(0.8,0.2,\ldots,0.1)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.5,0.4,\ldots,0.6),(0.6,0.5,\ldots,0.4),(0.5,0.2,\ldots,0.3)} \right) \right\}$$

$F_Q(e_2, r, 1)$
$$= \left\{ \left(\frac{(\mathbb{u}_1, q_1)}{(0.8,0.3,\ldots,0.4),(0.3,0.1,\ldots,0.5),(0.2,0.3,\ldots,0.4)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.7,0.3,\ldots,0.5),(0.3,0.2,\ldots,0.1),(0.4,0.3,\ldots,0.1)} \right) \right\}$$

$F_Q(e_1, p, 0)$
$$= \left\{ \left(\frac{(\mathbb{u}_1, q_1)}{(0.5,0.1,\ldots,0.2),(0.6,0.3,\ldots,0.4),(0.7,0.2,\ldots,0.6)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.6,0.5,\ldots,0.4),(0.5,0.1,\ldots,0.2),(0.6,0.1,\ldots,0.3)} \right) \right\}$$

$F_Q(e_1, r, 0)$
$$= \left\{ \left(\frac{(\mathbb{u}_1, q_1)}{(0.4,0.2,\ldots,0.1),(0.6,0.1,\ldots,0.3),(0.7,0.2,\ldots,0.4)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.4,0.3,\ldots,0.2),(0.1,0.2,\ldots,0.3),(0.5,0.4,\ldots,0.2)} \right) \right\}$$

$F_Q(e_2, p, 0)$
$$= \left\{ \left(\frac{(\mathbb{u}_1, q_1)}{(0.8,0.1,\ldots,0.5),(0.2,0.1,\ldots,0.4),(0.6,0.3,\ldots,0.1)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.7,0.3,\ldots,0.4),(0.6,0.7,\ldots,0.4),(0.2,0.1,\ldots,0.3)} \right) \right\}$$

$F_Q(e_2, r, 0)$
$$= \left\{ \left(\frac{(\mathbb{u}_1, q_1)}{(0.7,0.2,\ldots,0.3),(0.4,0.1,\ldots,0.6),(0.3,0.2,\ldots,0.1)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.5,0.6,\ldots,0.2),(0.3,0.4,\ldots,0.2),(0.4,0.1,\ldots,0.2)} \right) \right\}$$

The Q- neutrosophic soft expert multiset (F_Q, Z) is a parameterized family $\{F(e_i), i = 1, 2, \dots\}$ of all *QNSEMS* of $\mathbb{U}$ describing a collection of objects.

Definition 3.3. For two *QNSEMS*s (F_Q, G) and (H_Q, B) over $\mathbb{U}$, (F_Q, G) is called a neutrosophic soft expert subset of (H_Q, B) if

i. $B \subseteq G$,

ii. , $H_Q(\varepsilon)$ is Q-neutrosophic soft expert multisubset $F_Q(\varepsilon)$ for all $\varepsilon \in B$.

Example 3.4 Consider Example 3.2 where

$$G = \{(e_1, p, 1), (e_2, p, 1), (e_2, r, 0)\}$$
$$B = \{(e_1, p, 1), (e_2, r, 0)\}$$

Since B is a Q-neutrosophic soft expert multisubset of G, clearly $B \subseteq G$. Let (H_Q, B) and (F_Q, G) be defined as follows:

$(F_Q, G) =$
$$\left\{ \left[(e_1, p, 1), \left(\frac{(\mathbb{u}_1, q_1)}{(0.4, 0.3, \dots, 0.2), (0.5, 0.7, \dots, 0.2), (0.6, 0.1, \dots, 0.3)}\right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.7, 0.2, \dots, 0.4), (0.4, 0.6, \dots, 0.3), (0.2, 0.3, \dots, 0.4)}\right) \right] \right.$$
$$\left[(e_2, p, 1), \left(\frac{(\mathbb{u}_1, q_1)}{(0.7, 0.3, \dots, 0.6), (0.3, 0.2, \dots, 0.6), (0.8, 0.2, \dots, 0.1)}\right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.5, 0.4, \dots, 0.6), (0.6, 0.5, \dots, 0.4), (0.5, 0.2, \dots, 0.3)}\right) \right],$$
$$\left. \left[(e_2, r, 0), \left(\frac{(\mathbb{u}_1, q_1)}{(0.7, 0.2, \dots, 0.3), (0.4, 0.1, \dots, 0.6), (0.3, 0.2, \dots, 0.1)}\right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.5, 0.6, \dots, 0.2), (0.3, 0.4, \dots, 0.2), (0.4, 0.1, \dots, 0.2)}\right) \right] \right\}.$$

$(H_Q, B) =$
$$\left\{ \left[(e_1, p, 1), \left(\frac{(\mathbb{u}_1, q_1)}{(0.4, 0.3, \dots, 0.2), (0.5, 0.7, \dots, 0.2), (0.6, 0.1, \dots, 0.3)}\right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.7, 0.2, \dots, 0.4), (0.4, 0.6, \dots, 0.3), (0.2, 0.3, \dots, 0.4)}\right) \right], \right.$$
$$\left. \left[(e_2, r, 0), \left(\frac{(\mathbb{u}_1, q_1)}{(0.7, 0.2, \dots, 0.3), (0.4, 0.1, \dots, 0.6), (0.3, 0.2, \dots, 0.1)}\right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.5, 0.6, \dots, 0.2), (0.3, 0.4, \dots, 0.2), (0.4, 0.1, \dots, 0.2)}\right) \right] \right\}.$$
Therefore $(H_\eta, B) \subseteq (F_Q, G)$.

Definition 3.5. Two *QNSEMS*s (F_Q, G) and (H_Q, B) over $\mathbb{U}$ are said to be equal if (F_Q, G) is a *QNSEMS* subset of (H_Q, B) and (H_Q, B) is a *QNSEMS* subset of (F_Q, G).

Definition 3.6. Agree-Q-NSEMSs $(F_Q, G)_1$ over $\mathbb{U}$ is a *QNSEMS* subset of (F_Q, G) defined as

$$(F_Q, G)_1 = \{F_1(\Delta): \Delta \in E \times X \times \{1\}\}. \tag{6}$$

Example 3.7 Using our previous Example 3.2, the agree- *QNSEMS* $(F_Q, Z)_1$ over $\mathbb{U}$ is

$(F_Q, Z)_1$

$$= \left\{ \left[(e_1, p, 1), \left(\frac{(\mathbb{u}_1, q_1)}{(0.4, 0.3, ..., 0.2), (0.5, 0.7, ..., 0.2), (0.6, 0.1, ..., 0.3)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.7, 0.2, ..., 0.4), (0.4, 0.6, ..., 0.3), (0.2, 0.3, ..., 0.4)} \right) \right], \right.$$

$$\left[(e_1, r, 1), \left(\frac{(\mathbb{u}_1, q_1)}{(0.3, 0.2, ..., 0.5), (0.8, 0.1, ..., 0.4), (0.5, 0.6, ..., 0.2)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.8, 0.5, ..., 0.3), (0.5, 0.7, ..., 0.2), (0.3, 0.1, ..., 0.2)} \right) \right],$$

$$\left[(e_2, p, 1), \left(\frac{(\mathbb{u}_1, q_1)}{(0.7, 0.3, ..., 0.6), (0.3, 0.2, ..., 0.6), (0.8, 0.2, ..., 0.1)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.5, 0.4, ..., 0.6), (0.6, 0.5, ..., 0.4), (0.5, 0.2, ..., 0.3)} \right) \right],$$

$$\left. \left[(e_2, r, 1), \left(\frac{(\mathbb{u}_1, q_1)}{(0.8, 0.3, ..., 0.4), (0.3, 0.1, ..., 0.5), (0.2, 0.3, ..., 0.4)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.7, 0.3, ..., 0.5), (0.3, 0.2, ..., 0.1), (0.4, 0.3, ..., 0.1)} \right) \right] \right\}.$$

Definition 3.8. A disagree-*QNSEMS*s $(F_Q, G)_0$ over $\mathbb{U}$ is a *QNSEMS* subset of (F_Q, G) is defined as

$$(F_Q, G)_0 = \{F_0(\Delta): \Delta \in E \times X \times \{0\}\}. \tag{7}$$

Example 3.9 Using our previous Example 3.2, the disagree-*QNSEMS* $(F_Q, Z)_0$ over $\mathbb{U}$ is

$(F_Q, Z)_0$

$$= \left\{ \left[(e_1, p, 0), \left(\frac{(\mathbb{u}_1, q_1)}{(0.5, 0.1, ..., 0.2), (0.6, 0.3, ..., 0.4), (0.7, 0.2, ..., 0.6)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.6, 0.5, ..., 0.4), (0.5, 0.1, ..., 0.2), (0.6, 0.1, ..., 0.3)} \right) \right], \right.$$

$$\left[(e_1, r, 0), \left(\frac{(\mathbb{u}_1, q_1)}{(0.4, 0.2, ..., 0.1), (0.6, 0.1, ..., 0.3), (0.7, 0.2, ..., 0.4)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.4, 0.3, ..., 0.2), (0.1, 0.2, ..., 0.3), (0.5, 0.4, ..., 0.2)} \right) \right],$$

$$\left[(e_2, p, 0), \left(\frac{(\mathbb{u}_1, q_1)}{(0.8, 0.1, ..., 0.5), (0.2, 0.1, ..., 0.4), (0.6, 0.3, ..., 0.1)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.7, 0.3, ..., 0.4), (0.6, 0.7, ..., 0.4), (0.2, 0.1, ..., 0.3)} \right) \right],$$

$$\left. \left[(e_2, r, 0), \left(\frac{(\mathbb{u}_1, q_1)}{(0.7, 0.2, ..., 0.3), (0.4, 0.1, ..., 0.6), (0.3, 0.2, ..., 0.1)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.5, 0.6, ..., 0.2), (0.3, 0.4, ..., 0.2), (0.4, 0.1, ..., 0.2)} \right) \right] \right\}.$$

4. Basic operations on NSEMSs

Definition 4.1. The complement of a *QNSEMS* (F_Q, G) is

$$(F_Q, G)^c = \left(F_Q^{(c)}, \neg G\right)$$

such that $F^{\mu(c)}: \neg G \rightarrow QNSEMS(\mathbb{U})$ a mapping

$$F_Q^{(c)}(\Delta) = \left\{ D^i{}_{F_Q(\Delta)^{(c)}} = Y^i{}_{F_Q(\Delta)}, I^i{}_{F_Q(\Delta)^{(c)}} = \overline{1} - I^i{}_{F_Q(\Delta)}, Y^i{}_{F_Q(\Delta)^{(c)}} = D^i{}_{F_Q(\Delta)} \right\} \tag{8}$$

for each $\Delta \in E$.

Example 4.2. Using our previous Example 3.2 the complement of the *QNSEMS* F_Q denoted by $F_Q^{(c)}$ is given as follows:

$(F_Q, Z)^c$

$= \left\{ \left[(\neg e_1, p, 1), \left(\dfrac{(\mathbb{u}_1, q_1)}{(0.2,0.7,\dots,0.4),(0.2,0.3,\dots,0.5),(0.3,0.9,\dots,0.6)} \right), \left(\dfrac{(\mathbb{u}_1, q_2)}{(0.7,0.8,\dots,0.7),(0.3,0.4,\dots,0.4),(0.4,0.7,\dots,0.2)} \right) \right], \right.$

$\left[(\neg e_1, r, 1), \left(\dfrac{(\mathbb{u}_1, q_1)}{(0.5,0.8,\dots,0.3),(0.4,0.9,\dots,0.8),(0.2,0.4,\dots,0.5)} \right), \left(\dfrac{(\mathbb{u}_1, q_2)}{(0.3,0.5,\dots,0.8),(0.2,0.3,\dots,0.5),(0.2,0.9,\dots,0.3)} \right) \right],$

$\left[(\neg e_2, p, 1), \left(\dfrac{(\mathbb{u}_1, q_1)}{(0.6,0.7,\dots,0.7),(0.6,0.8,\dots,0.3),(0.1,0.8,\dots,0.8)} \right), \left(\dfrac{(\mathbb{u}_1, q_2)}{(0.6,0.6,\dots,0.5),(0.4,0.5,\dots,0.6),(0.3,0.8,\dots,0.5)} \right) \right],$

$\left[(\neg e_2, r, 1), \left(\dfrac{(\mathbb{u}_1, q_1)}{(0.4,0.7,\dots,0.8),(0.5,0.9,\dots,0.3),(0.4,0.7,\dots,0.2)} \right), \left(\dfrac{(\mathbb{u}_1, q_2)}{(0.5,0.7,\dots,0.7),(0.1,0.8,\dots,0.3),(0.1,0.7,\dots,0.4)} \right) \right],$

$\left[(\neg e_1, p, 0), \left(\dfrac{(\mathbb{u}_1, q_1)}{(0.2,0.9,\dots,0.5),(0.4,0.7,\dots,0.6),(0.6,0.8,\dots,0.7)} \right), \left(\dfrac{(\mathbb{u}_1, q_2)}{(0.4,0.5,\dots,0.6),(0.2,0.9,\dots,0.5),(0.3,0.9,\dots,0.6)} \right) \right]$

$\left[(\neg e_1, r, 0), \left(\dfrac{(\mathbb{u}_1, q_1)}{(0.1,0.8,\dots,0.4),(0.3,0.9,\dots,0.6),(0.4,0.8,\dots,0.7)} \right), \left(\dfrac{(\mathbb{u}_1, q_2)}{(0.2,0.7,\dots,0.4),(0.3,0.8,\dots,0.2),(0.2,0.6,\dots,0.5)} \right) \right],$

$\left[(\neg e_2, p, 0), \left(\dfrac{(\mathbb{u}_1, q_1)}{(0.5,0.9,\dots,0.8),(0.4,0.9,\dots,0.2),(0.1,0.7,\dots,0.6)} \right), \left(\dfrac{(\mathbb{u}_1, q_2)}{(0.4,0.7,\dots,0.7),(0.4,0.3,\dots,0.6),(0.3,0.9,\dots,0.2)} \right) \right],$

$\left. \left[(\neg e_2, r, 0), \left(\dfrac{(\mathbb{u}_1, q_1)}{(0.3,0.8,\dots,0.7),(0.6,0.9,\dots,0.4),(0.1,0.8,\dots,0.3)} \right), \left(\dfrac{(\mathbb{u}_1, q_2)}{(0.2,0.4,\dots,0.5),(0.2,0.6,\dots,0.3),(0.2,0.9,\dots,0.4)} \right) \right] \right\}$

Proposition 4.3. If (F_Q, G) is a *QNSEMS* over $\mathbb{U}$, then the properties below holds true.

1. $\left((F_Q, G)^c \right)^c = (F_Q, G)$

2. $\left((F_Q, G)_1 \right)^c = (F_Q, G)_0$

3. $\left((F_Q, G)_0 \right)^c = (F_Q, G)_1$

Proof. The proofs of the propositions are straightforward by using Definition 4.1, Definition 3.6 and Definition 3.8.

Definition 4.4. The union of two *QNSEMS*s (F_Q, G) and (K_Q, B) over $\mathbb{U}$, denoted by $(F_Q, G) \cup (K_Q, B)$ is the *QNSEMS*s (H_Q, C) such that $C = G \cup B$ and $\forall\, e \in C$,

$$(H_Q, C) = \begin{cases} max \left(D^i{}_{F_Q(e)}(m), D^i{}_{K_Q(e)}(m) \right) & if \;\; e \in G \cap B \\ min \left(I^i{}_{F_Q(e)}(m), I^i{}_{K_Q(e)}(m) \right) & if \;\; e \in G \cap B \\ min \left(Y^i{}_{F_Q(e)}(m), Y^i{}_{K_Q(e)}(m) \right) & if \;\; e \in G \cap B. \end{cases} \tag{9}$$

Example 4.5. Suppose that (F_Q, G) and (K_Q, B) are two *QNSEMS*s over $\mathbb{U}$, such that

$(F_Q, G) =$

$= \left\{ \left[(e_1, p, 0), \left(\dfrac{(\mathbb{u}_1, q_1)}{(0.7,0.3,\dots,0.5),(0.6,0.2,\dots,0.4),(0.4,0.5,\dots,0.1)} \right), \left(\dfrac{(\mathbb{u}_1, q_2)}{(0.4,0.3,\dots,0.5),(0.6,0.1,\dots,0.5),(0.4,0.3,\dots,0.1)} \right) \right], \right.$

$\left[(e_1, r, 0), \left(\dfrac{(\mathbb{u}_1, q_1)}{(0.5,0.1,\dots,0.7),(0.3,0.1,\dots,0.2),(0.4,0.3,\dots,0.2)} \right), \left(\dfrac{(\mathbb{u}_1, q_2)}{(0.3,0.1,\dots,0.2),(0.2,0.3,\dots,0.6),(0.5,0.4,\dots,0.2)} \right) \right],$

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$$\left[(e_2,p,0),\left(\frac{(\mathbb{u}_1,q_1)}{(0.8,0.4,\dots,0.6),(0.1,0.4,\dots,0.2),(0.5,0.2,\dots,0.6)}\right),\left(\frac{(\mathbb{u}_1,q_2)}{(0.7,0.3,\dots,0.4),(0.3,0.4,\dots,0.1),(0.2,0.3,\dots,0.4)}\right)\right],$$

$$\left[(e_2,r,0),\left(\frac{(\mathbb{u}_1,q_1)}{(0.6,0.3,\dots,0.4),(0.5,0.6,\dots,0.7),(0.1,0.3,\dots,0.4)}\right),\left(\frac{(\mathbb{u}_1,q_2)}{(0.5,0.6,\dots,0.2),(0.2,0.4,\dots,0.5),(0.3,0.1,\dots,0.5)}\right)\right]\Big\}$$

$(K_Q,B) =$

$$= \Big\{\left[(e_1,p,0),\left(\frac{(\mathbb{u}_1,q_1)}{(0.5,0.7,\dots,0.3),(0.4,0.3,\dots,0.7),(0.2,0.4,\dots,0.6)}\right),\left(\frac{(\mathbb{u}_1,q_2)}{(0.6,0.5,\dots,0.4),(0.5,0.1,\dots,0.2),(0.6,0.1,\dots,0.3)}\right)\right],$$

$$\left[(e_2,p,0),\left(\frac{(\mathbb{u}_1,q_1)}{(0.8,0.1,\dots,0.5),(0.2,0.1,\dots,0.4),(0.6,0.3,\dots,0.4)}\right),\left(\frac{(\mathbb{u}_1,q_2)}{(0.6,0.3,\dots,0.4),(0.6,0.3,\dots,0.4),(0.2,0.4,\dots,0.3)}\right)\right],$$

$$\left[(e_2,r,0),\left(\frac{(\mathbb{u}_1,q_1)}{(0.6,0.2,\dots,0.3),(0.4,0.1,\dots,0.6),(0.3,0.2,\dots,0.1)}\right),\left(\frac{(\mathbb{u}_1,q_2)}{(0.5,0.6,\dots,0.3),(0.3,0.4,\dots,0.7),(0.4,0.2,\dots,0.5)}\right)\right]\Big\}.$$

Then $(F_Q,G) \cup (K_Q,B) = (H_Q,C)$ where

(H_Q,C)

$$= \Big\{\left[(e_1,p,0),\left(\frac{(\mathbb{u}_1,q_1)}{(0.7,0.7,\dots,0.5),(0.4,0.2,\dots,0.4),(0.2,0.4,\dots,0.1)}\right),\left(\frac{(\mathbb{u}_1,q_2)}{(0.6,0.5,\dots,0.5),(0.5,0.1,\dots,0.2),(0.6,0.1,\dots,0.3)}\right)\right],$$

$$\left[(e_2,p,0),\left(\frac{(\mathbb{u}_1,q_1)}{(0.8,0.4,\dots,0.6),(0.1,0.1,\dots,0.2),(0.5,0.2,\dots,0.4)}\right),\left(\frac{(\mathbb{u}_1,q_2)}{(0.7,0.3,\dots,0.4),(0.3,0.3,\dots,0.1),(0.2,0.3,\dots,0.3)}\right)\right],$$

$$\left[(e_2,r,0),\left(\frac{(\mathbb{u}_1,q_1)}{(0.6,0.2,\dots,0.3),(0.4,0.1,\dots,0.6),(0.1,0.2,\dots,0.1)}\right),\left(\frac{(\mathbb{u}_1,q_2)}{(0.5,0.6,\dots,0.3),(0.2,0.4,\dots,0.5),(0.3,0.1,\dots,0.5)}\right)\right]\Big\}.$$

Proposition 4.6. If $(F_Q,G),(K_Q,B)$ and (H_Q,C) are three *QNSEMS*s over $\mathbb{U}$, then

i. $\left((F_Q,G) \cup (K_Q,B)\right) \cup (H_Q,C) = (F_Q,G) \cup \left((K_Q,B) \cup (H_Q,C)\right)$

ii $(F_Q,G) \cup (F_Q,G) \subseteq (F_Q,G).$

Proof. (i) and (ii) can be easily proved.

Definition 4.7. Suppose (F_Q,G) and (K_Q,B) are two *QNSEMS*s over the common universe $\mathbb{U}$. The intersection of (F_Q,G) and (K_Q,B) is $(F_Q,G) \cap (K_Q,B) = (P_Q,C)$ such that $C = G \cap B$ and $\forall\, e \in C$,

$$(P_Q,C) = \begin{cases} min\left(D^i{}_{F_Q(e)}(m), D^i{}_{K_Q(e)}(m)\right) & if\ \ e \in G \cap B \\ max\left(I^i{}_{F_Q(e)}(m), I^i{}_{K_Q(e)}(m)\right) & if\ \ e \in G \cap B \\ max\left(Y^i{}_{F_Q(e)}(m), Y^i{}_{K_Q(e)}(m)\right) & if\ \ e \in G \cap B. \end{cases} \tag{10}$$

Example 4.8. Suppose that (F_Q,G) and (K_Q,B) are two *QNSEMS*s over $\mathbb{U}$, such that

$(F_Q,G) =$

$$= \Big\{\left[(e_1,p,0),\left(\frac{(\mathbb{u}_1,q_1)}{(0.7,0.3,\dots,0.5),(0.6,0.2,\dots,0.4),(0.4,0.5,\dots,0.1)}\right),\left(\frac{(\mathbb{u}_1,q_2)}{(0.4,0.3,\dots,0.5),(0.6,0.1,\dots,0.5),(0.4,0.3,\dots,0.1)}\right)\right],$$

$$\left[(e_2,r,0),\left(\frac{(\mathbb{u}_1,q_1)}{(0.6,0.3,\dots,0.4),(0.5,0.6,\dots,0.7),(0.1,0.3,\dots,0.4)}\right),\left(\frac{(\mathbb{u}_1,q_2)}{(0.5,0.6,\dots,0.2),(0.2,0.4,\dots,0.5),(0.3,0.1,\dots,0.5)}\right)\right]\right\}.$$

$(K_Q,B) =$

$$= \left\{\left[(e_1,p,0),\left(\frac{(\mathbb{u}_1,q_1)}{(0.5,0.7,\dots,0.3),(0.4,0.3,\dots,0.7),(0.2,0.4,\dots,0.6)}\right),\left(\frac{(\mathbb{u}_1,q_2)}{(0.6,0.5,\dots,0.4),(0.5,0.1,\dots,0.2),(0.6,0.1,\dots,0.3)}\right)\right],\right.$$

$$\left[(e_2,p,0),\left(\frac{(\mathbb{u}_1,q_1)}{(0.8,0.1,\dots,0.5),(0.2,0.1,\dots,0.4),(0.6,0.3,\dots,0.4)}\right),\left(\frac{(\mathbb{u}_1,q_2)}{(0.6,0.3,\dots,0.4),(0.6,0.3,\dots,0.4),(0.2,0.4,\dots,0.3)}\right)\right],$$

$$\left.\left[(e_2,r,0),\left(\frac{(\mathbb{u}_1,q_1)}{(0.6,0.2,\dots,0.3),(0.4,0.1,\dots,0.6),(0.3,0.2,\dots,0.1)}\right),\left(\frac{(\mathbb{u}_1,q_2)}{(0.5,0.6,\dots,0.3),(0.3,0.4,\dots,0.7),(0.4,0.2,\dots,0.5)}\right)\right]\right\}.$$

Then $(F_Q,G)\cap(K_Q,B) = (P_Q,C)$ where

(P_Q,C)

$$= \left\{\left[(e_1,p,0),\left(\frac{(\mathbb{u}_1,q_1)}{(0.5,0.7,\dots,0.5),(0.4,0.3,\dots,0.7),(0.2,0.5,\dots,0.6)}\right),\left(\frac{(\mathbb{u}_1,q_2)}{(0.4,0.5,\dots,0.5),(0.5,0.1,\dots,0.5),(0.4,0.3,\dots,0.3)}\right)\right]\right.$$

$$\left.\left[(e_2,r,0),\left(\frac{(\mathbb{u}_1,q_1)}{(0.6,0.3,\dots,0.4),(0.4,0.6,\dots,0.7),(0.1,0.3,\dots,0.4)}\right),\left(\frac{(\mathbb{u}_1,q_2)}{(0.5,0.6,\dots,0.3),(0.2,0.4,\dots,0.7),(0.3,0.2,\dots,0.5)}\right)\right]\right\}.$$

Proposition 4.9. If $(F_Q,G),(K_Q,B)$ and (H_Q,C) are three *QNSEMS*s over $\mathbb{U}$, then

i. $\left((F_Q,G)\cap(K_Q,B)\right)\cap(H_Q,C) = (F_Q,G)\cap\left((K_Q,B)\cap(H_Q,C)\right).$

ii $(F_Q,G)\cap(F_Q,G) \subseteq (F_Q,G).$

Proof. The proofs are straightforward.

Proposition 4.10. If $(F_Q,G),(K_Q,B)$ and (H_Q,C) are three *QNSEMS*s over $\mathbb{U}$, then

i. $\left((F_Q,G)\cup(K_Q,B)\right)\cap(H_Q,C) = \left((F_Q,G)\cap(H_Q,C)\right)\cup\left((K_Q,B)\cap(H_Q,C)\right).$

ii. $\left((F_Q,G)\cap(K_Q,B)\right)\cup(H_Q,C) = \left((F_Q,G)\cup(H_Q,C)\right)\cap\left((K_Q,B)\breve{\cup}(H_Q,C)\right).$

Proof. The proofs can be easily obtained from Definition 4.4 and Definition 4.7.

5. AND and OR operations

Definition 5.1. If (F_Q,G) and (K_Q,B) are two *QNSEMS*s over $\mathbb{U}$, then $(F_Q,G)\text{AND}(K_Q,B)''$ is

$$(F_Q,G)\wedge(K_Q,B) = (H_Q,G\times B) \tag{11}$$

such that $H_Q(\alpha,\beta) = F_Q(\alpha)\cap K_Q(\beta)$ and of $(H_Q,G\times B)$ are as follows:

$$H_Q(\alpha,\beta)(m) = \begin{cases} min\left(D^i{}_{F_Q(e)}(m), D^i{}_{K_Q(e)}(m)\right) & if \ \ e \in G \cap B \\ min\left(I^i{}_{F_Q(e)}(m), I^i{}_{K_Q(e)}(m)\right) & if \ \ e \in G \cap B \\ max\left(Y^i{}_{F_Q(e)}(m), Y^i{}_{K_Q(e)}(m)\right) & if \ \ e \in G \cap B. \end{cases}$$

where $\forall \alpha \in G, \forall \beta \in B$.

Example 5.2. Suppose that $\left(F_Q, G\right)$ and $\left(K_Q, B\right)$ are two *QNSEMS*s over $\mathbb{U}$, such that

$\left(F_Q, G\right) =$

$= \left\{ \left[(e_1, p, 0), \left(\frac{(\mathbb{u}_1, q_1)}{(0.2,0.3, ..., 0.6), (0.2,0.1, ..., 0.8), (0.3,0.2, ..., 0.6)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.4,0.3, ..., 0.5), (0.6,0.1, ..., 0.5), (0.4,0.3, ..., 0.1)} \right) \right], \right.$

$\left. \left[(e_2, r, 0), \left(\frac{(\mathbb{u}_1, q_1)}{(0.5,0.3, ..., 0.4), (0.6,0.5, ..., 0.4), (0.2,0.4, ..., 0.3)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.5,0.6, ..., 0.2), (0.2,0.4, ..., 0.5), (0.3,0.1, ..., 0.5)} \right) \right] \right\}.$

$\left(K_Q, B\right)$

$= \left\{ \left[(e_1, p, 1), \left(\frac{(\mathbb{u}_1, q_1)}{(0.3,0.2, ..., 0.1), (0.5,0.2, ..., 0.3), (0.8,0.3, ..., 0.4)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.3,0.2, ..., 0.1), (0.5,0.2, ..., 0.3), (0.8,0.3, ..., 0.4)} \right) \right] \right\}$

Then $\left(F_Q, G\right) \wedge \left(K_Q, B\right) = \left(H_Q, G \times B\right)$ where

$\left(H_Q, G \times B\right) =$

$= \left\{ \left[(e_1, p, 0), (e_1, p, 1) \left(\frac{(\mathbb{u}_1, q_1)}{(0.2,0.2, ..., 0.6), (0.2,0.1, ..., 0.8), (0.3,0.2, ..., 0.6)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.3,0.2, ..., 0.5), (0.5,0.1, ..., 0.5), (0.4,0.3, ..., 0.4)} \right) \right], \right.$

$\left. \left[(e_2, r, 0), (e_1, p, 1) \left(\frac{(\mathbb{u}_1, q_1)}{(0.3,0.2, ..., 0.4), (0.5,0.2, ..., 0.8), (0.2,0.3, ..., 0.4)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.3,0.2, ..., 0.2), (0.2,0.2, ..., 0.5), (0.3,0.1, ..., 0.5)} \right) \right] \right\}.$

Definition 5.3. If $\left(F_Q, G\right)$ and $\left(K_Q, B\right)$ are two *QNSEMS*s over $\mathbb{U}$, then $\left(F_Q, G\right)$OR$\left(K_Q, B\right)$" is

$$\left(F_Q, G\right) \vee \left(K_Q, B\right) = \left(H_Q, G \times B\right) \tag{12}$$

such that $H_Q(\alpha, \beta) = F_Q(\alpha) \cap K_Q(\beta)$ and of $\left(H_Q, G \times B\right)$ are as follows:

$$H_Q(\alpha,\beta)(m) = \begin{cases} max\left(D^i{}_{F_Q(e)}(m), D^i{}_{K_Q(e)}(m)\right) & if \ \ e \in G \cap B \\ max\left(I^i{}_{F_Q(e)}(m), I^i{}_{K_Q(e)}(m)\right) & if \ \ e \in G \cap B \\ min\left(Y^i{}_{F_Q(e)}(m), Y^i{}_{K_Q(e)}(m)\right) & if \ \ e \in G \cap B. \end{cases} \tag{13}$$

where $\forall \alpha \in G, \forall \beta \in B$.

Example 5.4. Suppose that $\left(F_Q, G\right)$ and $\left(K_Q, B\right)$ are two *QNSEMS*s over $\mathbb{U}$, such that

$(F_Q, G) =$

$= \left\{ \left[(e_1, p, 0), \left(\dfrac{(\mathbb{u}_1, q_1)}{(0.2,0.3,\dots,0.6),(0.2,0.1,\dots,0.8),(0.3,0.2,\dots,0.6)} \right), \left(\dfrac{(\mathbb{u}_1, q_2)}{(0.4,0.3,\dots,0.5),(0.6,0.1,\dots,0.5),(0.4,0.3,\dots,0.1)} \right) \right], \right.$

$\left. \left[(e_2, r, 0), \left(\dfrac{(\mathbb{u}_1, q_1)}{(0.5,0.3,\dots,0.4),(0.6,0.5,\dots,0.4),(0.2,0.4,\dots,0.3)} \right), \left(\dfrac{(\mathbb{u}_1, q_2)}{(0.5,0.6,\dots,0.2),(0.2,0.4,\dots,0.5),(0.3,0.1,\dots,0.5)} \right) \right] \right\}.$

(K_Q, B)

$= \left\{ \left[(e_1, p, 1), \left(\dfrac{(\mathbb{u}_1, q_1)}{(0.3,0.2,\dots,0.1),(0.5,0.2,\dots,0.3),(0.8,0.3,\dots,0.4)} \right), \left(\dfrac{(\mathbb{u}_1, q_2)}{(0.3,0.2,\dots,0.1),(0.5,0.2,\dots,0.3),(0.8,0.3,\dots,0.4)} \right) \right] \right\}$

Then $(F_Q, G) \vee (K_Q, B) = (H_Q, G \times B)$ where

$(H_Q, G \times B) =$

$= \left\{ \left[(e_1, p, 0), (e_1, p, 1) \left(\dfrac{(\mathbb{u}_1, q_1)}{(0.3,0.3,\dots,0.1),(0.5,0.2,\dots,0.3),(0.8,0.3,\dots,0.4)} \right), \left(\dfrac{(\mathbb{u}_1, q_2)}{(0.4,0.3,\dots,0.1),(0.6,0.2,\dots,0.3),(0.8,0.3,\dots,0.1)} \right) \right], \right.$

$\left. \left[(e_2, r, 0), (e_1, p, 1) \left(\dfrac{(\mathbb{u}_1, q_1)}{(0.5,0.3,\dots,0.1),(0.6,0.5,\dots,0.3),(0.8,0.4,\dots,0.3)} \right), \left(\dfrac{(\mathbb{u}_1, q_2)}{(0.5,0.6,\dots,0.1),(0.5,0.4,\dots,0.3),(0.8,0.3,\dots,0.4)} \right) \right] \right\}.$

Proposition 5.5. If (F_Q, G) and (K_Q, B) are two *QNSEMS*s over $\mathbb{U}$, then

$i. \left((F_Q, G) \wedge (K_Q, B) \right)^c = (F_Q, G)^c \vee (K_Q, B)^c$

$ii. \left((F_Q, G) \vee (K_Q, B) \right)^c = (F_Q, G)^c \wedge (K_Q, B)^c$

Proof The proofs are straightforward from Definition 5.1 and Definition 5.3.

6. An Application of QNSEMSs

In this section, we will now present an application in architecture of *QNSEMS* theory to illustrate that this concept can be successfully applied to decision-making problems with uncertain information. The following algorithm is suggested to solve a *QNSEMS* based decision making problem below.

In the preventing water permeability process, the cold and hot cycles are prevented from damaging the structure. When choosing membrane types for insulation, it is important to determine the sealing thickness on the surface to be used. Therefore, this method gives the best type of membrane. Let us assume that the membrane application outputs used in for insulation structures are taken by a few experts at certain time intervals. So, let us take the samples at three different timings in a day (in 09:30, 14:30 and 19:30). Ezgi construction will make the membrane purchase. Two types of membrane (alternatives) $\mathbb{U} = \{\mathbb{u}_1, \mathbb{u}_2\}$ with two types of qualifications $Q = \{q_1, q_2\}$ and there are two parameters $E = \{e_1, e_2\}$ where the parameters e_i $(i = 1,2)$ stand for "hot" and "cold" respectively. Let $X = \{p, q\}$ be a set of experts. After a good application process, the experts construct the Q-NSEMS below.

$(F_Q, Z) =$

$$\left\{ \left[(e_1, p, 1), \left(\frac{(\mathbb{u}_1, q_1)}{(0.3,0.1,0.4),(0.2,0.1,0.5),(0.5,0.2,0.6)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.4,0.2,0.3),(0.7,0.1,0.6),(0.3,0.2,0.6)} \right), \left(\frac{(\mathbb{u}_2, q_1)}{(0.5,0.3,0.4),(0.2,0.1,0.8),(0.8,0.2,0.1)} \right), \left(\frac{(\mathbb{u}_2, q_2)}{(0.9,0.3,0.4),(0.2,0.1,0.8),(0.4,0.2,0.1)} \right) \right], \right.$$

$$\left[(e_1, q, 1), \left(\frac{(\mathbb{u}_1, q_1)}{(0.4,0.2,0.5),(0.3,0.1,0.2),(0.6,0.3,0.4)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.5,0.3,0.2),(0.8,0.2,0.4),(0.5,0.3,0.2)} \right), \left(\frac{(\mathbb{u}_2, q_1)}{(0.6,0.3,0.8),(0.3,0.2,0.1),(0.5,0.4,0.3)} \right), \left(\frac{(\mathbb{u}_2, q_2)}{(0.6,0.3,0.8),(0.3,0.2,0.1),(0.5,0.4,0.3)} \right) \right],$$

$$\left[(e_2, p, 1), \left(\frac{(\mathbb{u}_1, q_1)}{(0.6,0.4,0.2),(0.3,0.1,0.4),(0.8,0.2,0.5)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.8,0.3,0.4),(0.2,0.1,0.5),(0.4,0.3,0.1)} \right), \left(\frac{(\mathbb{u}_2, q_1)}{(0.8,0.3,0.2),(0.3,0.1,0.4),(0.2,0.1,0.4)} \right), \left(\frac{(\mathbb{u}_2, q_2)}{(0.4,0.3,0.7),(0.3,0.1,0.4),(0.5,0.3,0.2)} \right) \right],$$

$$\left[(e_2, q, 1), \left(\frac{(\mathbb{u}_1, q_1)}{(0.5,0.2,0.4),(0.3,0.2,0.5),(0.6,0.1,0.3)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.6,0.4,0.7),(0.5,0.3,0.2),(0.6,0.2,0.4)} \right), \left(\frac{(\mathbb{u}_2, q_1)}{(0.6,0.5,0.4),(0.1,0.3,0.2),(0.6,0.2,0.3)} \right), \left(\frac{(\mathbb{u}_2, q_2)}{(0.8,0.2,0.3),(0.2,0.1,0.4),(0.3,0.4,0.5)} \right) \right],$$

$$\left[(e_1, p, 0), \left(\frac{(\mathbb{u}_1, q_1)}{(0.5,0.1,0.7),(0.4,0.2,0.3),(0.5,0.4,0.1)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.8,0.2,0.3),(0.2,0.1,0.4),(0.3,0.4,0.5)} \right), \left(\frac{(\mathbb{u}_2, q_1)}{(0.5,0.2,0.6),(0.3,0.4,0.1),(0.2,0.3,0.1)} \right), \left(\frac{(\mathbb{u}_2, q_2)}{(0.5,0.2,0.3),(0.4,0.1,0.2),(0.2,0.1,0.4)} \right) \right],$$

$$\left[(e_1, q, 0), \left(\frac{(\mathbb{u}_1, q_1)}{(0.7,0.1,0.4),(0.3,0.2,0.1),(0.4,0.2,0.5)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.6,0.5,0.4),(0.4,0.2,0.1),(0.8,0.2,0.6)} \right), \left(\frac{(\mathbb{u}_2, q_1)}{(0.9,0.4,0.5),(0.2,0.1,0.3),(0.6,0.2,0.3)} \right), \left(\frac{(\mathbb{u}_2, q_2)}{(0.6,0.5,0.4),(0.1,0.3,0.2),(0.6,0.2,0.3)} \right) \right],$$

$$\left[(e_2, p, 0), \left(\frac{(\mathbb{u}_1, q_1)}{(0.6,0.5,0.7),(0.3,0.5,0.4),(0.6,0.3,0.4)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.5,0.2,0.3),(0.2,0.1,0.3),(0.4,0.3,0.5)} \right), \left(\frac{(\mathbb{u}_2, q_1)}{(0.6,0.3,0.4),(0.1,0.2,0.4),(0.5,0.3,0.2)} \right), \left(\frac{(\mathbb{u}_2, q_2)}{(0.3,0.1,0.4),(0.2,0.1,0.5),(0.5,0.2,0.6)} \right) \right],$$

$$\left. \left[(e_2, q, 0), \left(\frac{(\mathbb{u}_1, q_1)}{(0.3,0.1,0.2),(0.4,0.1,0.3),(0.5,0.2,0.6)} \right), \left(\frac{(\mathbb{u}_1, q_2)}{(0.7,0.2,0.4),(0.4,0.3,0.6),(0.5,0.1,0.6)} \right), \left(\frac{(\mathbb{u}_2, q_1)}{(0.7,0.3,0.5),(0.2,0.4,0.3),(0.5,0.2,0.3)} \right), \left(\frac{(\mathbb{u}_2, q_2)}{(0.5,0.3,0.2),(0.8,0.2,0.4),(0.5,0.3,0.2)} \right) \right] \right\}.$$

Tables 1 presents the agree-*QNSEMS* while Table 2 presents the disagree-*Q-NSEMS* by using the mean of each *QNSEMS*.

The following algorithm may be used to choose the most qualified membrane to preventing water permeability.

   Input the $QNSEMS \left(F_Q, Z \right)$.

1. Compute the

$$\text{agree-}QNSES\ (\mathbb{u}, q) = \left| max\{D^i\} - min\{I^i\} - min\{Y^i\} \right|$$

and

$$\text{disagree-}QNSES\ (\mathbb{u}, q) = \left| min\{D^i\} - max\{I^i\} - max\{Y^i\} \right|$$

2. Find the agree-*QNSEMS* and disagree-*QNSEMS*.

3. Calculate $c_j = \sum_i (\mathbb{u}, q)_{ij}$ for agree-*QNSEMS*.

4. Calculate $k_j = \sum_i (\mathbb{u}, q)_{ij}$ for disagree-*QNSEMS*.

5. Determine $s_j = \left| c_j - k_j \right|$.

6. Determine r, for which $s_r = max s_j$. If there is has more than a one value of r, then the membrane can have alternative choices.

Table 1: Agree- *QNSEMS*.

	$(\mathbb{u}_1, q_1)$	$(\mathbb{u}_1, q_2)$	$(\mathbb{u}_2, q_1)$	$(\mathbb{u}_2, q_2)$
$(e_1, p, 1)$	0.1	0.3	0.6	0.7
$(e_2, p, 1)$	0.5	0.6	0.5	0.2
$(e_1, q, 1)$	0.3	0.3	0.3	0.3
$(e_2, q, 1)$	0.2	0.1	0.2	0.4
$C_j = \sum_i (\mathbb{u}, q)_{ij}$	$c_1 = 1.1$	$c_2 = 1.3$	$c_3 = 1.6$	$c_4 = 1.6$

Table 2: Disagree- *QNSEMS*.

	$(\mathbb{u}_1, q_1)$	$(\mathbb{u}_1, q_2)$	$(\mathbb{u}_2, q_1)$	$(\mathbb{u}_2, q_2)$
$(e_1, p, 0)$	0.7	0.7	0.9	0.4
$(e_2, p, 0)$	0.9	0.6	0.6	0.8
$(e_1, q, 0)$	0.6	0.7	0.6	0.6
$(e_2, q, 0)$	0.5	0.5	0.6	0.1
$k_j = \sum_i (\mathbb{u}, q)_{ij}$	$k_1 = 2.7$	$k_2 = 2.5$	$k_3 = 2.7$	$k_4 = 1.9$

Table 3: $s_j = |c_j - k_j|$

j	$V \times Q$	c_j	k_j	s_j
1	$(\mathbb{u}_1, q_1)$	1.1	2.7	1.6
2	$(\mathbb{u}_1, q_2)$	1.3	2.5	1.2
3	$(\mathbb{u}_2, q_1)$	1.6	2.7	1.1
4	$(\mathbb{u}_2, q_2)$	1.6	1.9	0.3

From Tables 1 and 2 we are able to calculate the values of $s_j = c_j - k_j$ as in Table 3.

As can be seen, the maximum score is the score 1.6, shown in the above for the $\mathbb{u}_1$. Hence the best decision for the experts is to select membrane $\mathbb{u}_2$ followed by.

7.Comparison Analysis

A *Q*-neutrosophic soft expert model gives more precision, flexibility and compatibility compared to the classical, fuzzy and/or intuitionistic fuzzy models.

Table4: Comparison of fuzzy soft set to other variants

	Fuzzy soft expert	*Q*-Fuzzy soft	Multi *Q*-fuzzy soft expert	Q-Neutrosophic soft set	Q-Neutrosophic Soft Expert	*Q*-Neutrosophic soft expert multiset
Methods	Alkhazaleh and Salleh [22]	Adam and Hassan [33]	Adam and Hassan [28]	Sahin et al. [29]	Hassan et.al[27]	Proposed Method
Domain	Universe of discourse	Universe of discourse	Universe of discourse	Universe of discourse	Universe of discourse	Universe of discourse
Co-domain	[0,1]	[0,1]	[0,1]	$[0,1]^3$	$[0,1]^3$	$[0,1]^3$
True	Yes	Yes	Yes	Yes	Yes	Yes
Falsity	No	No	No	Yes	Yes	Yes
Indeterminacy	No	No	No	Yes	Yes	Yes
Expert	Yes	No	Yes	No	Yes	Yes
Q	No	Yes	Yes	No	Yes	Yes
Membership valued	Single valued	Single valued	multi-valued	Single valued	Single-valued	Multi-valued

The feasibility and effectiveness of the proposed decision-making approach are verified by a comparison analysis using neutrosophic soft expert multiset decision method, with those methods used by Sahin et al. [29], Adam and Hassan [28,33] and Alkhazaleh and Salleh [22], as given in Table 4, based on the same example as in Section 4. The ranking order results obtained are consistent with those in [16,22,28,29,33].

8. Conclusion

We have introduced the concept of a *Q* -neutrosophic soft expert set along with its operations of equality, union, intersection, subset, OR, and AND. It is shown that this proposed concept is more inclusive by taking into account the membership of falsity and indeterminacy, expert, neutrosophy and *Q* -fuzzy. Thus the proposed approach is shown to be useful in handling realistic uncertain problems. Finally an application of the constructed algorithm to solve a decision-making problem is provided. This new extension will provide a significant addition to existing theories for handling indeterminacy, and spurs more developments of further research and pertinent applications.

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Chapter Nine

An outperforming approach for multi-criteria decision-making problems with interval-valued Bipolar neutrosophic sets

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ABSTRACT

In this chapter, a different outperforming access for MCDM problems is recommended to approach positions pointing with in each cluster of numbers in the absolute system interval and unequitable a definitive number among a bipolar neutrosophic set. Mostly, the procedures of inter-valued bipolar neutrosophic sets and their associated characters are imported. Formerly certain outperforming similarities for inter-valued bipolar neutrosophic numbers (IVBNNs) are described depend on ELECTRE, and the characters of the outperforming similarities are farther considered definitely. Furthermore, depend on the outperforming similarities of IVBNSs, a ranking approach is advanced that one may clarify MCDM problems.

Key Words: Neutrosophic sets, bipolar neutrosophic sets, Interval-valued bipolar neutrosophic sets, Multicriteria decision-making, Outranking method.

1 Introduction

So as to overcome different kinds of confusions, the seminal theory of fuzzy sets [1] has been proposed in 1965 by Zadeh; meanwhile, it has been tested strongly in different areas [2]. Nonetheless, in a few cases it's ambitious to define the rate of membership of a FS along a certain value. Because of this reason, Turksen introduced the IVFSs [3]. Afterwards, Atanassov [4, 5] investigated the notion of intuitionistic fuzzy sets (IFSs) to run-over the illiteracy of nonmembership degrees. Until now, IFS outmoded worldwide tested to figure out MCDM problems [6–8] in areas like medical images [9], pattern recognition [10], edge detection and game theory [11-12] and image fusion [13]. IFSs were afterwards approached to IVIFSs [14], along with to IVIFSs with triangular IFNs [15]. So to carry out these problems where one is doubtful in showing their choice respecting phenomenon in a DMP, hesitant fuzzy sets were developed [16-17] between 2009-2010. Moreover, defined

more expansions have been suggested [18–20], and mechanisms along IFNs from a few unsual groups' decision-making surveys have been advanced [21-22]. In spite of the fact that the concept of FSs has hold advanced and concluded, it couldn't cope along all kinds of confusions, like imprecise and incompatible information, in accurate DMPs. For instance [23], during an authority provide the idea around a particular description, the one may estimate a certain probability that the description is accurate is 0.5, the rate of inaccurate description is 0.6, and the probability that the one isn't sure is 0.2. The concept of neutrosophic logic and neutrosophic sets [24-25] has been developed in 1995 by Smarandache. Since then, it is applied to various areas, such as decision making problems[38-43]. The NS is a set wither all member of the universe has a rate of accuracy, uncertainity and falsity and that deceit in]0-, 1[, the abnormal system interval [26]. Obviously, this's the extension to the normal interval [0, 1] as in the IFS. Additionally, the confusion present here, such that, indefinity cause, is separate of accuracy and falsity values, when the integrated confusion is reliant of the rates of belongingness and non-belongingness in IFSs [27]. Furthermore, regarding the mentioned example around expert description, it can be shown as $x(0\colon 5;\ 0\colon 2;\ 0\colon 6)$ by NSs.

2. Preliminary
In the subsection, we present defined notions containing neutrosophic sets, bipolar neutrosophic
sets and interval valued bipolar neutrosophic sets.

2.1 Neutrosophic Sets [28]
Let E be a universe. A neutrosophic sets A over E is defined by

$$A = \left\{ \left\langle x, T_A(x), I_A(x), F_A(x) \right\rangle : x \in E \right\}$$

where $T_A(x), I_A(x)$ ve $F_A(x)$ are called truth-membership function, indeterminacy-membership
function and falsity-membership function, respectively. They are respectively defined by

$$T_A(x) : E \rightarrow \left]0^-, 1^+\right[\quad , I_A(x) : E \rightarrow \left]0^-, 1^+\right[\ , \ F_A(x) : E \rightarrow \left]0^-, 1^+\right[$$

such that

$$0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+.$$

2.2. Single Valued Neutrosophic Set [29]

Let E be a universe. A(SVN-set) overE is a neutrosophic set over E, but the truth-membership function, indeterminacy-membership function and falsity-membership function are respectively defined by

$$T_A(x):E \to [0,1], I_A(x):E \to [0,1] , \ F_A(x):E \to [0,1]$$

such that $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

2.3. Bipolar Neutrosophic Set [30]

A BNS A in X is defined as an object of the form

$$A = \left\{ \left\langle x, T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x), \right\rangle : x \in X \right\}$$

where

$$T^+(x), I^+(x), F^+(x): X \to [0,1], T^-(x), I^-(x), F^-(x): X \to [-1,0]$$

The positive membership degree $T^+(x), I^+(x), F^+(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ corresponding to a bipolarneutrosophic set A and the negative membership degree $T^-(x), I^-(x), F^-(x)$ denotes the truthmembership, indeterminate membership and false membership of an element $x \in X$ to someimplicit counter-property corresponding to a bipolar neutrosophic set A.

2.4. Interval Valued Bipolar Neutrosophic Set [31]

An IVBNS A in X is defined as an object of the form

$$A = \left\langle \left[T_L^+(x), T_R^+(x)\right], \left[I_L^+(x), I_R^+(x)\right], \left[F_L^+(x), F_R^+(x)\right], \left[T_L^-(x), T_R^-(x)\right], \left[I_L^-(x), I_R^-(x)\right], \left[F_L^-(x), F_R^-(x)\right] \right\rangle$$

where $T_L^+, T_R^+, I_L^+, I_R^+, F_L^+, F_R^+ : X \to [0,1]$ and $T_L^-, T_R^-, I_L^-, I_R^-, F_L^-, F_R^- : X \to [-1,0]$.

2.5. The Operations for IVBNNs [31]

Let $A_1 = \left\langle \left[T_{1L}^+, T_{1R}^+\right], \left[I_{1L}^+, I_{1R}^+\right], \left[F_{1L}^+, F_{1R}^+\right], \left[T_{1L}^-, T_{1R}^-\right], \left[I_{1L}^-, I_{1R}^-\right], \left[F_{1L}^-, F_{1R}^-\right] \right\rangle$ and

$A_2 = \left\langle \left[T_{2L}^+, T_{2R}^+\right], \left[I_{2L}^+, I_{2R}^+\right], \left[F_{2L}^+, F_{2R}^+\right], \left[T_{2L}^-, T_{2R}^-\right], \left[I_{2L}^-, I_{2R}^-\right], \left[F_{2L}^-, F_{2R}^-\right] \right\rangle$ be two interval

valued bipolar neutrosophic number. Then the operations for IVBNNs are defined as below;

i.

$$\lambda A_1 = \left\langle \begin{array}{l} \left[1-\left(1-T_L^+\right)^\lambda, 1-\left(1-T_R^+\right)^\lambda\right], \left[\left(I_L^+\right)^\lambda, \left(I_R^+\right)^\lambda\right], \left[\left(F_L^+\right)^\lambda, \left(F_R^+\right)^\lambda\right], \\ \left[-\left(-T_L^-\right)^\lambda, -\left(-T_R^-\right)^\lambda\right], \left[-\left(-I_L^-\right)^\lambda, -\left(-I_R^-\right)^\lambda\right], \left[-(1-\left(1-(-F_L^-)\right)^\lambda), -(1-\left(1-(-F_R^-)\right)^\lambda)\right] \end{array} \right\rangle$$

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ii.

$$A_1^\lambda = \left\langle \begin{array}{l} \left[\left(T_L^+\right)^\lambda, \left(T_R^+\right)^\lambda\right], \left[1-\left(1-I_L^+\right)^\lambda, 1-\left(1-I_R^+\right)^\lambda\right], \left[1-\left(1-F_L^+\right)^\lambda, 1-\left(1-F_R^+\right)^\lambda\right], \\ \left[-(1-\left(1-(-T_L^-)\right)^\lambda), -(1-\left(1-(-T_R^-)\right)^\lambda)\right], \left[-\left(-I_L^-\right)^\lambda, -\left(-I_R^-\right)^\lambda\right], \left[-\left(-F_L^-\right)^\lambda, -\left(-F_R^-\right)^\lambda\right] \end{array} \right\rangle$$

iii.

$$A_1 + A_2 = \left\langle \begin{array}{l} \left[T_{1L}^+ + T_{2L}^+ - T_{1L}^+ . T_{2L}^+, T_{1R}^+ + T_{2R}^+ - T_{1R}^+ . T_{2R}^+\right], \left[I_{1L}^+ . I_{2L}^+, I_{1R}^+ . I_{2R}^+\right], \\ \left[F_{1L}^+ . F_{2L}^+, F_{1R}^+ . F_{2R}^+\right], \left[-T_{1L}^- . T_{2L}^-, -T_{1R}^- . T_{2R}^-\right], \\ \left[-(-I_{1L}^- - I_{2L}^- - I_{1L}^- . I_{2L}^-), -(-I_{1R}^- - I_{2R}^- - I_{1R}^- . I_{2R}^-)\right], \left[-(-F_{1L}^- - F_{2L}^- - F_{1L}^- . F_{2L}^-), -(-F_{1R}^- - F_{2R}^- - F_{1R}^- . F_{2R}^-)\right] \end{array} \right\rangle$$

iv.

$$A_1 + A_2 = \left\langle \begin{array}{l} \left[T_{1L}^+ . T_{2L}^+, T_{1R}^+ . T_{2R}^+\right], \left[I_{1L}^+ + I_{2L}^+ - I_{1L}^+ . I_{2L}^+, I_{1R}^+ + I_{2R}^+ - I_{1R}^+ . I_{2R}^+\right], \\ \left[F_{1L}^+ + F_{2L}^+ - F_{1L}^+ . F_{2L}^+, F_{1R}^+ + F_{2R}^+ - F_{1R}^+ . F_{2R}^+\right], \left[-(-T_{1L}^- - T_{2L}^- - T_{1L}^- . T_{2L}^-), -(-T_{1R}^- - T_{2R}^- - T_{1R}^- . T_{2R}^-)\right], \\ \left[I_{1L}^- . I_{2L}^-, I_{1R}^- . I_{2R}^-\right], \left[F_{1L}^- . F_{2L}^-, F_{1R}^- . F_{2R}^-\right] \end{array} \right\rangle$$

where $\lambda > 0$.

3 Outranking relations of IVBNNs

A pseudo-criterion is a criterion including preference and indifference thresholds. The definition of the pseudo-criterion[32-33] was provided by Roy. The pseudo-criterion is a function g_j such that it is linked to two criterion functions, $q_j(.)$ and $p_j(.)$. In other words, a function g_j and threshold functions together constitute the pseudo-criterion. It should satisfy conditions as follow: $\forall (b,b') \in B \times B$, $g_j(b) \geq g_j(b'), g_j(b) + p_j(g_j(b'))$ and $g_j(b) + q_j(g_j(b'))$ are unreducing monotone functions of $g_j(b')$, and $p_j(g_j(b')) \geq q_j(g_j(b'))$ for all $b \in B$, where $q_j(g_j(b'))$ and $p_j(g_j(b'))$ are the greatest and smallest performance difference, for which the situation of indifference holds on to criterion g_j between two actions b and b' . According to the definition, for the pseudo-criterion between two IVBNs we can give Definition 3.1 as follows.

3.1. Relation Between Two IVBNs

Given two IVBNs a and b, where
$$a = \left\langle \left[\mu_{aL}^+, \mu_{aR}^+\right], \left[v_{aL}^+, v_{aR}^+\right], \left[\omega_{aL}^+, \omega_{aR}^+\right], \left[\mu_{aL}^-, \mu_{aR}^-\right], \left[v_{aL}^-, v_{aR}^-\right], \left[\omega_{aL}^-, \omega_{aR}^-\right] \right\rangle \text{ and}$$

$$b = \left\langle \left[\mu_{bL}^+, \mu_{bR}^+\right], \left[v_{bL}^+, v_{bR}^+\right], \left[\omega_{bL}^+, \omega_{bR}^+\right], \left[\mu_{bL}^-, \mu_{bR}^-\right], \left[v_{bL}^-, v_{bR}^-\right], \left[\omega_{bL}^-, \omega_{bR}^-\right] \right\rangle . \text{Let}$$

$$g(a) = \left[\left(\mu_{aL}^+ + \mu_{aR}^+\right) + \left(\mu_{aL}^- + \mu_{aR}^-\right) \right] - \left[\left(v_{aL}^+ + v_{aR}^+\right) + \left(v_{aL}^- + v_{aR}^-\right) \right] - \left[\left(\omega_{aL}^+ + \omega_{aR}^+\right) + \left(\omega_{aL}^- + \omega_{aR}^-\right) \right],$$

$$g(b) = \left[\left(\mu_{bL}^+ + \mu_{bR}^+\right) + \left(\mu_{bL}^- + \mu_{bR}^-\right) \right] - \left[\left(v_{bL}^+ + v_{bR}^+\right) + \left(v_{bL}^- + v_{bR}^-\right) \right] - \left[\left(\omega_{bL}^+ + \omega_{bR}^+\right) + \left(\omega_{bL}^- + \omega_{bR}^-\right) \right],$$

$$g(T_{ab}) = \left[\left(\mu_{aL}^+ + \mu_{aR}^+\right) + \left(\mu_{aL}^- + \mu_{aR}^-\right) \right] - \left[\left(\mu_{bL}^+ + \mu_{bR}^+\right) + \left(\mu_{bL}^- + \mu_{bR}^-\right) \right],$$

$$g(I_{ab}) = \left[\left(v_{bL}^+ + v_{bR}^+\right) + \left(v_{bL}^- + v_{bR}^-\right) \right] - \left[\left(v_{aL}^+ + v_{aR}^+\right) + \left(v_{aL}^- + v_{aR}^-\right) \right] \text{ and}$$

$$g(F_{ab}) = \left[\left(\omega_{bL}^+ + \omega_{bR}^+\right) + \left(\omega_{bL}^- + \omega_{bR}^-\right) \right] - \left[\left(\omega_{aL}^+ + \omega_{aR}^+\right) + \left(\omega_{aL}^- + \omega_{aR}^-\right) \right] . \text{ Assume that}$$

$$g(a,b) = g(a) - g(b) = g(T_{ab}) + g(I_{ab}) + g(F_{ab}).$$

Let's assume that p and q are respectively are the option criterion and the indifference criterion. Relations exist between the two IVBNs are defined as follows:

(1) In case that p<g(a,b), suddenly a is fully approved to b, symbolized as P(a,b) or $a > s^b$.

(2) In case that $q < g(a,b) \le p$, suddenly a is defectively approved to b, symbolized as Q (a,b) or $a > w^b$.

(3) In case that $-q < g(a,b) \le q$, suddenly a is indifferent to b, symbolized as I(a,b) or $a > \iota^b$.

Obviously, the common three circumstances with regard to similarities among two IVNNs a and b are a active, weak and indifference influence similarity. Here, a actively influences b in case that it's fully preferred to b. Unless, a defectively influences b.

Numerical Example. Let's assume that p and q are respectively the preference criterion and the indifference criterion, and let p = 0.4, q = 0.2.

(1) If $a = \left\langle [0.5, 0.8], [0.3, 0.4], [0.1, 0.2], [-0.2, -0.1], [-0.5, -0.2], [-0.8, -0.7] \right\rangle$ and

$b = \left\langle [0.5, 0.6], [0.4, 0.5], [0.2, 0.4], [-0.4, -0.2], [-0.9, -0.5], [-0.7, -0.6] \right\rangle$

- are two IVBNs, then g(a,b)=0.5>p. Accordingly, a and b fascinate first condition, namely, a is defectively chosen to b.

(2) If $a = \left\langle [0.3, 0.8], [0.3, 0.9], [0.1, 0.3], [-0.7, -0.6], [-0.6, -0.2], [-0.4, -0.1] \right\rangle$ and

$b = \left\langle [0.4, 0.7], [0.6, 0.8], [0.3, 0.4], [-0.9, -0.5], [-0.4, -0.3], [-0.8, -0.1] \right\rangle$

are two IVBNs, then $q < g(a,b) = 0.3 \le p$. Accordingly, a and b fascinate second condition, namely, a is defectively chosen, to b.

(3) If $a = \langle [0.3,0.9],[0.1,0.8],[0.2,0.5],[-0.8,-0.7],[-0.5,-0.1],[-0.4,-0.3] \rangle$ and

$b = \langle [0.3,0.8],[0.3,0.9],[0.1,0.3],[-0.7,-0.6],[-0.6,-0.2],[-0.4,-0.1] \rangle$

are two IVBNs, then –q<g(a,b)=-0.1<q. Accordingly, a and b satisfy I(a, b), namely,a is indifferent to b.

Property 1 Assume that a, b and c are IVBNs, if $a > s^b$ and $b > s^c$, then $a > s^c$:

Proof In accordance with Definition 3.1, if $a > s^b$ then g(a,b)>p, that is, g(a)-g(b)>p. Similarly, if $b > s^c$ then g(b)-g(c)>p.Therefore, g(a)-g(b)+g(b)-g(c)>2p. Thus, g(a)-g(c)>p.Based on Definition 3.1, $a > s^c$ is obtained and, thus, we have easily proven that the property 1 is true.

Property 2 Assume that a, b and c are IVBNs, formerly the conclusions can be obtained as follows.

(1) The active influence similarities are classified into:

(1a) There is no reflexivity, that is, $\forall a \in IVBNs, a \not> s^a$;

(1b) There is no symmetry, namely, $\forall a,b \in IVBNs, a > s^b \Rightarrow b \not> s^a$;

(1c)There is transitivity, namely, $\forall a,b,c \in IVBNs, a > s^b, b > s^c \Rightarrow a > s^c$.

(2) The defective influence similarities are classified into:

(2a) There is no reflexivity, namely, $\forall a \in IVBNs, a \not> w^a$;

(2b) There is no symmetry, that is, $\forall a,b \in IVBNs, a > w^b \Rightarrow b \not> w^a$;

(2c)There is no transitivity, namely, $\exists a,b,c \in IVBNs, a > w^b, b > w^c \not\Rightarrow a > w^c$.

(3) The indifference similarities are classified into:

(3a) There is reflexivity, that is, $\forall a \in IVBNs, a \sim \iota^a$;

(3b) There is symmetry, namely, $\forall a,b \in IVBNs, a \sim \iota^b \Rightarrow b \sim \iota^a$;

(3c)There is no transitivity, namely, $\exists a,b,c \in IVBNs, a \sim \iota^b, b \sim \iota^c \not\Rightarrow a \sim \iota^c$.

g(a)-g(a)=0<q<p , $a \not> s^a$, $a \not> w^a$, $a \sim \iota^a$ holds in accordance with 3.1.

Hereby, (1a), (2a) and (3a) are accurate.

Likewise, in case that $a > s^b$, formerly $g(a) - g(b) > p \Rightarrow g(b) - g(a) < -p$. Accordingly,

$b \not> s^a$ and (1b) is correct, in the fact (2b) and (3b). In accordance with Property 1 and Property 2, (1c) is correct. In addition to this, (2c) and (3c) have to be proven.

3.3. Binary Comparison of Two Sets

Let options A and B be clusters of IVBNs, $A = \{a_1, a_2, a_3, ..., a_m\}$, and

$B = \{b_1, b_2, b_3, ..., b_m\}, a_i, b_j \in IVBNs(i, j = 1, 2, ..., m)$. Sets A and B can be binary compared using the following notations.:

The number of the criteria $P(a_i, b_j)$ is represented by $n_P(A, B)$,

The number of the criteria $Q(a_i, b_j)$ is represented by $n_Q(A, B)$,

The number of the approach $I(a_i, b_j) \wedge (g(a_i, b_j) > 0)$ is represented by $n_I(A, B)$,

When two options A and B are considered, the succeeding expressions give an idea about the outperforming similarity [34]:

(1) A outperforms B, symbolized as AOB, i.e, the investigator may present adequate reasons to decision-makers in relation to the proposition "A is somewhat approximate B".

(2) A doesn't outperform B, i.e, there is no reason enough to validate proposition ''A is somewhat approximate B'', but it doesn't mean that ''B is somewhat approximate A'', where the symbols "I" and "R" represent indifference and incomparability, respectively. Moreover, There are two cases for the preference relation. They're active option and defective option, symbolized as $A > s^B$ and $A > w^B$ jointly.

Depend on [35], the rules of influence were build up thusly.

3.4. Number of Criteria

Let's conclude that there are m criteria in total. Then:

(1) $A > s^B$: (a) for no threshold B is closely approved to A;

 (b) If the number of approach for that B is defectively approved in similar to A is lesser or fit the number of approach for that A is closely approved to B; and (c) and in case that the number of approach, for that the achievement of B is greater than that of A, is closely lesser to the number of approach for that the achievement of A is greater than such of option B.

$$A > s^B \Leftrightarrow \{n_P(B, A) = 0 \text{ ve } n_Q(B, A) \leq n_P(A, B) \text{ ve } n_Q(B, A) + n_I(B, A) < n_I(A, B) + n_Q(A, B) + n_P(A, B).$$

(2) $A > w^B$: (a) in case that concealed by no approach B is closely approved to A; (b) in case that the number of approach, for that the achievement of B is preferable to such of A, is closely lesser to the number of criteria for which the achievement of A is preferable to that of option B; (c) in case that the further position needed for the similar A S B carry on isn't documented; and (d) in case that there's a particular threshold for that B is closely approved to A, formerly A is closely approved to B in somewhat limited of the approach.

$$A > w^B \Leftrightarrow \{n_P(B, A) = 0 \text{ ve } n_Q(B, A) + n_I(B, A) < n_I(A, B) + n_Q(A, B) + n_P(A, B) \text{ ve not } A > s^B \text{veya } n_P(B, A) = 1 \text{ ve } n_P(A, B) \geq m / 2.$$

Numerical Example. consider that the option criterion is p = 0.4 and the nonchalance criterion is q = 0.2. For alternatives B₁ and B₂, the decision maker must decide according to three criteria. The achievement of all threshold in similar to all option is as shown in the following:

$$B_1 = [B_{11} \ B_{12} \ B_{13}] = \begin{cases} \langle a, [0.3, 0.9], [0.1, 0.8], [0.2, 0.5], [-0.8, -0.7], [-0.5, -0.1], [-0.4, -0.3] \rangle, \\ \langle b, [0.3, 0.8], [0.3, 0.9], [0.1, 0.3], [-0.7, -0.6], [-0.6, -0.2], [-0.4, -0.1] \rangle, \\ \langle c, [0.4, 0.5], [0.5, 0.6], [0.3, 0.5], [-0.4, -0.2], [-0.9, -0.5], [-0.7, -0.6] \rangle \end{cases}$$

$$B_2 = [B_{21} \ B_{22} \ B_{23}] = \begin{cases} \langle a, [0.2, 0.8], [0.3, 0.6], [0.3, 0.6], [-0.3, -0.2], [-0.6, -0.2], [-0.5, -0.4] \rangle, \\ \langle b, [0.4, 0.7], [0.5, 0.7], [0.2, 0.3], [-0.2, -0.1], [-0.8, -0.3], [-0.9, -0.8] \rangle, \\ \langle c, [0.4, 0.7], [0.6, 0.8], [0.3, 0.4], [-0.9, -0.5], [-0.4, -0.3], [-0.8, -0.1] \rangle \end{cases}$$

In accordance with 3.1,

$g(B_{11}, B_{21}) = -1$, $g(B_{21}, B_{11}) = 1 > p$, $g(B_{22}, B_{12}) = 2.4 > p$ ve $g(B_{13}, B_{23}) = 1.9 > p$. Hence,

$P(B_{21}, B_{11}), P(B_{22}, B_{12})$ ve $P(B_{13}, B_{23})$. That is, the relationship between alternatives

B_1 and B_2 is that with regard to the former two criteria, B_2 is strictly preferred to B_1, while

B_1 is closely

approved to B_2 with respect to the third threshold. For the pair (B_1, B_2) ;

$n_P(B_1, B_2) = 1, n_Q(B_1, B_2) = 0, n_I(B_1, B_2) = 0, n_P(B_2, B_1) = 2, n_Q(B_2, B_1) = 0$ ve $n_I(B_2, B_1) = 0$.

As reported by definition 3.4, $B_2 \succ w^{B_1}$; consequently B_2 is defectively afflicted by B_1.

For ease of presentation, let $n_s(A_i)$ show the number that has active influence relation

among the optione A_i , for which $A_i \succ s^{A_k}$ between the pair (A_i, A_k) (k=1,2,3,...,n) .

Similarly, $n_w(A_i)$ represents the number for which $A_i \succ w^{A_k}$,for k=1,2,3,...,n.

Definition 3.6. The rule of the outperforming similarity among (A_i, A_k) are build up as :

(1) If $n_s(A_i) > n_s(A_j)$, then it implies A_i outranking A_j , denoted $A_i \succ A_j$.

(2) If $n_s(A_i) = n_s(A_j)$, and $n_w(A_i) > n_w(A_j)$, then $A_i \succ A_j$.

(3) If $n_s(A_i) = n_s(A_j)$, $n_w(A_i) = n_w(A_j)$ and $A_i \succ s^{A_j}$, then $A_i \succ A_j$.

Afterwards, the sequence among a partly or full order of choices is settled.

4 An outperforming method for MCDM with IVBNSs

In this section, an outperforming approach for MCDM problems where the outperforming tests of IVBNSs are utilizing is suggested.

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Suppose such the MCDM rating/choosing problem among IVBNSs amount to n options $A = \{A_1, A_2, A_3, ..., A_n\}$ and all option is calculated by m criteria $C = \{c_1, c_2, c_3, ..., c_m\}$. The calculation of all option according to their threshold is turn into in the IN decision matrix $R = (A_{ij})_{n \times m}$, where $A_{ij} = \left\langle \mu^+_{A_{ij}}, v^+_{A_{ij}}, \omega^+_{A_{ij}}, \mu^-_{A_{ij}}, v^-_{A_{ij}}, \omega^-_{A_{ij}} \right\rangle$ is a threshold value, symbolized as IVBNN, where the positive membership degree $\mu^+_{A_{ij}}, v^+_{A_{ij}}, \omega^+_{A_{ij}}$ shows the accuracy, uncertainity and falsity-membership function such the option A_i fascinates the criterion c_j, and the negative membership degree $\mu^-_{A_{ij}}, v^-_{A_{ij}}, \omega^-_{A_{ij}}$ shows the truth-membership, indeterminacy-membership and falsity-membership function that the option A_i fascinates the criterion c_j. Let the option threshold for each criterion $P = \{p_1, p_2, p_3, ..., p_m\}$ and the indifference threshold $Q = \{q_1, q_2, q_3, ..., q_m\}$.

Let's see the given procedure used to rate and choose the best option(s) is summarized.

Procedure 1 Let G be performance matrix and c_j each criterion of every alternative A_i. Compute G. Then, the performance value of A_i on c_j is symbolized as $g(A_i)_j$. By utilizing 3.1, G can be evaluated:

$$G = \begin{bmatrix} g(A_1)_1 & g(A_1)_2 & \cdots & g(A_1)_m \\ g(A_2)_1 & g(A_2)_2 & \cdots & g(A_2)_m \\ \cdots & \cdots & \cdots & \cdots \\ g(A_n)_1 & g(A_n)_2 & \cdots & g(A_n)_m \end{bmatrix}$$

Procedure 2 Let D be difference matrix. Compute D. Then,

The difference value $g(A_i, A_k)_j$ describes the performance difference between two A_i and A_k on c_j. Conforming to the score function the rate of all option as to every c_j criterion, $g(A_i, A_k)_j = g(A_i)_j - g(A_k)_j (i, k = 1, 2, .., n; j = 1, 2, ..., m)$. Consequently, D can be formulated as:

$$D = \begin{bmatrix} g(A_1, A_1)_1 & g(A_1, A_1)_2 & \cdots & g(A_1, A_1)_m \\ g(A_1, A_2)_1 & g(A_1, A_2)_2 & \cdots & g(A_1, A_2)_m \\ \cdots & \cdots & \cdots & \cdots \\ g(A_n, A_n)_1 & g(A_n, A_n)_2 & \cdots & g(A_n, A_n)_m \end{bmatrix}$$

Procedure 3 Achieve the binary relationship between two the IVBNNs as to c_j for A_i.

Definitely, the difference value $g(A_i, A_k)_j$ should be analyzed along the option criterion p_j and nonchalance criterion q_j to regulate the similarities. By 3.1, in case that

$g(A_i,A_k)_j > p_j$, then $A_{ij} > s^{A_{kj}}$. Moreover, if $q_j < g(A_i,A_k)_j < p_j$, then $A_{ij} > w^{A_{kj}}$ and if $-q_j < g(A_i,A_k)_j < q_j$, then $A_{ij} > \iota^{A_{kj}}$.

Procedure 4 By utilizing definition 3.3, count the number of outperforming relations $n_P(A_i,A_k), n_Q(A_i,A_k)$ ve $n_I(A_i,A_k)$ for the pair-wise (A_i,A_k) respecting every criteria. Definitely, any

pair-wise (A_i,A_k) $n_P(A_i,A_k)$ represents the number when $A_{ij} > s^{A_{kj}}$ for j=1,2,...,m . Likewise, $n_Q(A_i,A_k)$ represents the number when $A_{ij} \sim \iota^{A_{kj}}$ and $n_I(A_i,A_k)$ represents the number when $A_{ij} \sim \iota^{A_{kj}}$ and $g(A_i,A_k)_j > 0$ for j=1,2,...,m .

Procedure 5 As reported by definition 3.4, complete the outperforming relations between (A_i,A_k) .

Procedure 6 Count $n_s(A_i)$ and $n_w(A_i)$ for alternative A_i .

Procedure 7 As reported by definition 3.6 choose the alternative(s) with the best outperforming relations and the largest $n_s(A_i)$. If two or more have the maximum of $n_s(A_i)$, then compare $n_w(A_i)$. consequently, the alternatives with the best outperforming relations are extracted.

Procedure 8 Rerun procedure 4–7 for the halting alternatives as far as the remainder is empty.

Procedure 9 Choose the alternative(s). The whole or limited order for A_i is finally settled.

5. MCDMP

In this section, to denote the utilization of the requested decision-making approach and its capability, a numerical examples of MCDMPs along options are furnished.

5.1. Numerical Example

A MCDMP familiarized from Refs. [36-37] will be utilized. There is a group along four desirable options to lend capital: B_1, B_2, B_3 and B_4 . The lending company must take a decision according to the succeeding three criteria: c_1, c_2 and c_3 . The option criterion $P = \{0.2, 0.2, 0.2\}$ and the nonchalance criterion $Q = \{0.1, 0.1, 0.1\}$. The four desirable options are to be calculated concealed by the upon three approach, and the calculated rates are to be turn into IVBNNs, as indicated in the succeeding IN decision matrix D:

	C_1
B_1	$\langle[0.4,0.5],[0.1,0.4],[0.1,0.6],[-0.3,-0.2],[-0.7,-0.3],[-0.5,-0.4]\rangle$
B_2	$\langle[0.3,0.7],[0.3,0.5],[0.3,0.5],[-0.4,-0.3],[-0.8,-0.6],[-0.6,-0.5]\rangle$
B_3	$\langle[0.5,0.8],[0.2,0.7],[0.4,0.5],[-0.9,-0.6],[-0.6,-0.2],[-0.3,-0.2]\rangle$
B_4	$\langle[0.1,0.2],[0.2,0.7],[0.1,0.3],[-0.6,-0.3],[-0.9,-0.4],[-0.7,-0.2]\rangle$

	C_2
B_1	$\langle[0.2,0.8],[0.1,0.7],[0.1,0.4],[-0.9,-0.8],[-0.6,-0.2],[-0.5,-0.2]\rangle$
B_2	$\langle[0.1,0.7],[0.1,0.3],[0.2,0.3],[-0.6,-0.2],[-0.4,-0.2],[-0.9,-0.5]\rangle$
B_3	$\langle[0.1,0.5],[0.2,0.8],[0.2,0.4],[-0.9,-0.8],[-0.5,-0.4],[-0.8,-0.7]\rangle$
B_4	$\langle[0.1,0.2],[0.7,0.8],[0.1,0.6],[-0.6,-0.5],[-0.7,-0.4],[-0.6,-0.4]\rangle$

	C_3
B_1	$\langle[0.1,0.5],[0.1,0.4],[0.1,0.3],[-0.6,-0.3],[-0.8,-0.4],[-0.5,-0.3]\rangle$
B_2	$\langle[0.2,0.3],[0.1,0.5],[0.4,0.6],[-0.6,-0.2],[-0.9,-0.8],[-0.6,-0.3]\rangle$
B_3	$\langle[0.2,0.8],[0.1,0.7],[0.1,0.2],[-0.6,-0.5],[-0.7,-0.6],[-0.8,-0.7]\rangle$
B_4	$\langle[0.1,0.6],[0.4,0.7],[0.7,0.8],[-0.9,-0.8],[-0.9,-0.6],[-0.6,-0.3]\rangle$

Procedure 1 Compute G:

$$G = \begin{bmatrix} 1.1 & -0.5 & 0.8 \\ 1.2 & 1.1 & 0.7 \\ -0.7 & -0.3 & 1.6 \\ 0.3 & -0.9 & -1.2 \end{bmatrix}$$

Procedure 2 Compute D.

The difference value $g(B_i, B_k)_j$ between the two alternatives B_i and B_k on c_j fascinates $g(B_i, B_k)_j = g(B_i)_j - g(B_k)_j (i, k = 1, 2, .., n; j = 1, 2, ..., m)$. Finally, D can be constructed as indicated in Table 1.

Table 1 The performance difference of projects on each criterion

	c_1	c_2	c_3
(B_1, B_1)	0	0	0
(B_1, B_2)	-0.1	-1.6	0.1
(B_1, B_3)	1.8	-0.2	-0.8
(B_1, B_4)	0.8	0.4	2
(B_2, B_1)	0.1	1.6	-0.1
(B_2, B_2)	0	0	0
(B_2, B_3)	1.9	1.4	-0.9
(B_2, B_4)	0.9	2	1.9
(B_3, B_1)	-1.8	0.2	0.8
(B_3, B_2)	-1.9	-1.4	0.9
(B_3, B_3)	0	0	0
(B_3, B_4)	-1	0.6	2.8
(B_4, B_1)	-0.8	-0.4	-2
(B_4, B_2)	-0.9	-2	-1.9
(B_4, B_3)	1	-0.6	-2.8
(B_4, B_4)	0	0	0

Procedure 3 Achieve the binary similarity among the two IVBNNs on c_j for A_i. Analyze $g(B_i, B_k)_j$ along p_j and q_j, and formely, the conclusions can be indicated in Table 2.

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Table 2 The relationship between each pair-wise on the criteria

	c_1	c_2	c_3
(B_1, B_1)	~I	~I	~I
(B_1, B_2)	~I	<S	~I
(B_1, B_3)	>S	<W	<S
(B_1, B_4)	>S	>S	>S
(B_2, B_1)	~I	>S	~I
(B_2, B_2)	~I	~I	~I
(B_2, B_3)	>S	>S	<S
(B_2, B_4)	>S	>S	>S
(B_3, B_1)	<S	>W	>S
(B_3, B_2)	<S	<S	>S
(B_3, B_3)	~I	~I	~I
(B_3, B_4)	<S	>S	>S
(B_4, B_1)	<S	<S	<S
(B_4, B_2)	<S	<S	<S
(B_4, B_3)	>S	<S	<S
(B_4, B_4)	~I	~I	~I

Procedure 4 Evaluate the number of all outperforming similarity of every A_i along the other options for every c_j , as seen in Table 3.

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Table 3 The relation numbers between pair-wise for all the criteria

	$n_P(B_i,B_k)$	$n_Q(B_i,B_k)$	$n_I(B_i,B_k)$
(B_1,B_1)	0	0	0
(B_1,B_2)	0	0	1
(B_1,B_3)	1	0	0
(B_1,B_4)	3	0	0
(B_2,B_1)	1	0	1
(B_2,B_2)	0	0	0
(B_2,B_3)	2	0	0
(B_2,B_4)	3	0	0
(B_3,B_1)	1	1	0
(B_3,B_2)	1	0	0
(B_3,B_3)	0	0	0
(B_3,B_4)	2	0	0
(B_4,B_1)	0	0	0
(B_4,B_2)	0	0	0
(B_4,B_3)	1	0	0
(B_4,B_4)	0	0	0

Procedure 5
As reported by 3.4, determine the outperforming relations as seen in Table 4.

Table 4 The outranking relation matrix of the projects on the criteria

	B_1	B_2	B_3	B_4
B_1	I			>s
B_2	>s	I	>w	>s
B_3			I	>w
B_4				I

Procedure 6 Count $n_s(B_i)$ for all alternative B_i as seen in Table 5.

Table 5 The relation numbers between each project and the others

	$n_s(B_i)$	$n_w(B_i)$
B_1	1	0
B_2	2	1
B_3	0	1
B_4	0	0

Procedure 7 As reported by 3.6 choose the best outperforming alternative(s).

It is obvious to see that B_2 is the best alternative.

Procedure 8 Rerun procedure 4 to 7 for the halting alternatives as far as the remainder is empty.

The method is repeatedly rerunned, and finally B_1, B_3 and B_4 are settled.

Procedure 9 According to the above procedures, select the best alternatives and the final order is as certained as $B_2 \succ B_1 \succ B_3 \succ B_4$.

6. Conclusion

Interval valuable bipolar neutrosophic sets are a new branch of neutrosophic sets. There are some problems in real scientific and engineering utilizations such that these problems contain undetermined, incomplete and inconsistent information. Interval-valued bipolar neutrosophic sets can be applied to overcome such problems. In this chapter, an approach was presented to figure out MCDM problems using IVBNSs. As a result, an outperforming way to solve MCDM problems using IVBNSs was developed depend on the ELEKTRE IV. Hence, a few outperforming relations for IVBNSs were introduced, and also the properties related to the outperforming relations were reviewed categorically. On the other hand, two examples were utilized to denote the application of the method.

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Chapter Ten

A New Approach Distance Measure of Bipolar Neutrosophic Sets and Its Application to Multiple Criteria Decision Making

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Abstract

A bipolar neutrosophic set (BNS) is an instance of a single- valued neutrosophic set. To do this, we firstly propose distance measure between two BNSs is defined by the full consideration of positive membership function and negative membership function for the forward and backward differences. Then the similarity measure, the entropy measure and the index of distance are also presented. Then, two examples are shown to verify the feasibility of the proposed method. Finally, the decision results of different similarity measures demonstrate the practicality and effectiveness of the developed method in this paper.

Keywords: Single-valued neutrosophic sets, Single-valued bipolar neutrosophic sets, decision making.

1. Introduction

The MCDM is an important part of modern decision science and relate to many complex factors, such as economics, psychological behavior, ideology, military and so on. In many real-life decisions making problems can be modelling with fuzzy set theory (Zadeh 1965) and intuitionistic fuzzy set theory (Atanassov 1986). Because of membership functions, these theories have some disadvantages and cannot modelling MCDM problems. Based on the theories, Smarandache (1998) developed the neutrosophic set theory which overcomes the disadvantage of fuzzy set theory and intuitionistic fuzzy set theory which independently has a truth-membership degree, an indeterminacy-membership degree and a falsity-membership degree. Also Lee (2000, 2009) bipolar fuzzy set developed to modelling some real problems to some implicit counter-property which has positive membership degree and negative membership degree. Many research treating imprecision and uncertainty have been developed and studied. Since then, it is applied to various areas, such as decision making problems (Athar 2014, Aydogdu 2015, Broumi & Smarandache 2013, Balasubramanian, Prasad & Arjunan 2015, Broumi, Deli & Smarandache 2014, Chen, Li, Ma &Wang 2014, Chen 2014, Deli & Broumi 2015, Broumi, Bakali, Talea and Smarandache 2017, Broumi, Bakali, Talea and Smarandache 2016, Broumi, Smarandache, Talea and Bakali 2016, Broumi, Bakali, Talea and Smarandache 2018, Karaaslan 2016, Majumder 2012, Majumdar & Samanta 2014, Smarandache 2005, Santhi & Shyamala 2015, Saeid 2009, Shen, Xu & Xu 2016, Sahin, Deli & Ulucay 2016, Wang, Wang, Zhang

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Recently, Deli et al. (2015) proposed bipolar neutrosophic set theory and their operations based on bipolar fuzzy set theory and neutrosophic set theory. A bipolar neutrosophic set theory have the positive membership degrees $T^+(x), I^+(x), F^+(x)$ and the negative membership degrees $T^-(x), I^-(x), F^-(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$. Then Deli et al. (2016), Şahin et al. (2016) and Uluçay et al. (2016) presented some different similarity measure and applied to multi-attribute decision making problems.

Clustering plays an important part in analyzing the real world, such as pattern recognition, data mining, machine learning and so on. Over the past few decades, researchers has been used clustering method in many fields studies (Gua, Xia, Sengür & Polat 2016, Gua & Sengür 2015, Koundal, Gupta & Singh 2016, Roy 1991, Wu & Chen 2011).

The rest of paper is organized as follows. In Sect. 2, we review basic concepts about neutrosophic sets and bipolar neutrosophic sets. In Sect. 3, the notions of the distance measure, the similarity measure, the entropy measure and the index of distance are introduced. In Sect. 4, two illustrate examples are given to show the effectiveness of the new distance measure applied in clustering and decision making. In Sect. 5, a comparison analysis and discussion is conducted between the proposed approach and other existing methods, in order to verify its feasibility and effectiveness. Finally, the conclusions are drawn.

2. Preliminaries

Definition 2.1.(Smarandache 1998) Let X be a universe of discourse. Then a neutrosophic set is defined as:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in X\},$$

which is characterized by a truth-membership function $T_A: X \to]0^-1^+[$, an indeterminacy-membership function $I_A: X \to]0^-1^+[$ and a falsity-membership function $F_A: X \to]0^-1^+[$.

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \leq$ sup $T_A(x) \leq$ sup $I_A(x) \leq$ sup $F_A(x) \leq 3^+$

Definition 2.2. (Wang, Smarandache, Zhang & Sunderraman 2010) Let X be a universe of discourse. Then a single valued neutrosophic set (SVNS) is defined as:

$$A_{NS} = \{\langle x, F_A(x), T_A(x), I_A(x) \rangle : x \in X\},$$

which is characterized by a truth-membership function $T_A: X \rightarrow [0,1]$, an indeterminacy-membership function $I_A: X \rightarrow [0,1]$ and a falsity-membership function $F_A: X \rightarrow [0,1]$.

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0 \leq \sup T_A(x) \leq \sup I_A(x) \leq \sup F_A(x) \leq 3$.

Definition 2.3. (Deli et al. 2015) Let X be a universe of discourse. A bipolar neutrosophic set A_{BNS} in X is defined as an object of the form

$$A_{BNS} = \left\{ \left\langle x, T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x) \right\rangle : x \in X \right\},$$

where $T^+, I^+, F^+ : X \rightarrow [1,0]$ and $T^-, I^-, F^- : X \rightarrow [-1,0]$.

The positive membership degree $T^+(x), I^+(x), F^+(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ corresponding to a bipolar neutrosophic set A_{BNS} and the negative membership degree $T^-(x), I^-(x), F^-(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ to some implicit counter-property corresponding to a bipolar neutrosophic set A_{BNS}.

Set- theoretic operations, for two bipolar neutrosophic set

$$A_{BNS} = \left\{ \left\langle x, T_1^+(x), I_1^+(x), F_1^+(x), T_1^-(x), I_1^-(x), F_1^-(x) \right\rangle : x \in X \right\}$$

and

$$B_{BNS} = \left\{ \left\langle x, T_2^+(x), I_2^+(x), F_2^+(x), T_2^-(x), I_2^-(x), F_2^-(x) \right\rangle : x \in X \right\}$$

are given as;
1. The subset; $A_{BNS} \subseteq B_{BNS}$ if and only if

$$T^+_1(x) \leq T_2^+(x) \; I^+_1(x) \leq I_2^+(x), F^+_1(x) \geq F_2^+(x),$$

and

$$T^-_1(x) \geq T_2^-(x), \; I^-_1(x) \geq I^-_2(x), \; F_1^-(x) \leq F_2^-(x)$$

for all $x \in X$.

2. $A_{BNS} = B_{BNS}$ if and only if,

$$T^+_1(x) = T_2^+(x), \; I^+_1(x) = I_2^+(x), \; F^+_1(x) = F_2^+(x),$$

and

$$T^-_1(x) = T_2^-(x), \; I^-_1(x) = I^-_2(x), \; F_1^-(x) = F_2^-(x)$$

for all $x \in X$.

3. The complement of A_{BNS} is denoted by A_{BNS}^o and is defined by

$$T_{A^c}^{+}(x) = \{1^+\} - T_A^{+}(x), \quad I_{A^c}^{+}(x) = \{1^+\} - I_A^{+}(x), \quad F_{A^c}^{+}(x) = \{1^+\} - F_A^{+}(x)$$

and

$$T_{A^c}^{-}(x) = \{1^-\} - T_A^{-}(x), \quad I_{A^c}^{-}(x) = \{1^-\} - I_A^{-}(x), \quad F_{A^c}^{-}(x) = \{1^-\} - F_A^{-}(x),$$

for all $x \in X$.

4. The intersection

$$(A_{BNS} \cap B_{BNS})(x) = \begin{cases} \left\langle x, \min(T_1^{+}(x), T_2^{+}(x)), \dfrac{I_1^{+}(x) + I_2^{+}(x)}{2}, \max((F_1^{+}(x), F_2^{+}(x)), \max(T_1^{-}(x), T_2^{+}(x)), \right. \\ \left. \dfrac{I_1^{-}(x) + I_2^{-}(x)}{2}, \min((F_1^{-}(x), F_2^{-}(x)) \right\rangle : x \in X \end{cases}$$

5. The union

$$(A_{BNS} \cup B_{BNS})(x) = \begin{cases} \left\langle x, \max(T_1^{+}(x), T_2^{+}(x)), \dfrac{I_1^{+}(x) + I_2^{+}(x)}{2}, \min((F_1^{+}(x), F_2^{+}(x)), \right. \\ \left. \min(T_1^{-}(x), T_2^{+}(x)), \dfrac{I_1^{-}(x) + I_2^{-}(x)}{2}, \max((F_1^{-}(x), F_2^{-}(x)) \right\rangle : x \in X \end{cases}.$$

Definition 2.9. (Deli et al. 2015) Let $\tilde{a}_1 = \langle\, T_1^+, I_1^+, F_1^+, T_1^-, I_1^-, F_1^- \,\rangle$ and $\tilde{a}_2 = \langle\, T_2^+, I_2^+, F_2^+, T_2^-, I_2^-, F_2^- \,\rangle$ be two bipolar neutrosophic number. Then the operations for BNNs are defined as below;

    i. $\lambda \tilde{a}_1 = \langle\, 1 - (1 - T_1^+)^\lambda, (I_1^+)^\lambda, (F_1^+)^\lambda, -(-T_1^-)^\lambda, -(-I_1^-)^\lambda, -(1 - (1 - (-F_1^-))^\lambda) \,\rangle$

    ii. $\tilde{a}_1^\lambda = \langle\, (T_1^+)^\lambda, 1 - (1 - I_1^+)^\lambda, 1 - (1 - F_1^+)^\lambda, -(1 - (1 - (-T_1^-))^\lambda), -(-I_1^-)^{\,\lambda}, -(-F_1^-)^\lambda \,\rangle$

    iii. $\tilde{a}_1 + \tilde{a}_2 = \langle\, T_1^+ + T_2^+ - T_1^+ T_2^+, I_1^+ I_2^+, F_1^+ F_2^+, -T_1^- T_2^-, -(-I_1^- - I_2^- - I_1^- I_2^-), -(-F_1^- - F_2^- - F_1^- F_2^-) \,\rangle$

    iv. $\tilde{a}_1 . \tilde{a}_2 = \langle\, T_1^+ T_2^+, I_1^+ + I_2^+ - I_1^+ I_2^+, F_1^+ + F_2^+ - F_1^+ F_2^+, -(-T_1^- - T_2^- - T_2^- T_2^-), -I_1^- I_2^-, -F_1^- F_2^- \,\rangle$

where $\lambda > 0$.

3. Distance Measure of Bipolar Neutrosophic Sets

In this section, we defined distance measure two between bipolar neutrosophic sets that are based on Clustering method by extending the studies in (Huang 2016).

Definition3.1. For two bipolar neutrosophic A_1 and A_2 in a universe of discourse $X = \{x_1 x_2, \dots, x_n \quad\}$ which are denoted by $A_1 = \langle x_i, T_1^+(x_i), I_1^+(x_i), F_1^+(x_i), T_1^-(x_i), I_1^-(x_i), F_1^-(x_i) \rangle$ and $A_2 = \langle x_i, T_2^+(x_i), I_2^+(x_i), F_2^+(x_i), T_2^-(x_i), I_2^-(x_i), F_2^-(x_i) \rangle$ .The bipolar neutrosophic weighted distance measure are defined by

$$
d_\lambda(A_1, A_2)
= \left[\sum_{j=1}^{n} \omega_j \left(\sum_{i=1}^{4} \beta_i \, \varphi_i(x_j) \right)^\lambda \right]^{\frac{1}{\lambda}} \tag{1}
$$

where $\lambda > 0$, $\beta_i \in [0,1]$ and $\sum_{i=1}^{4} \beta_i = 1$, $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$

$$
\varphi_1(x_j) = \left(\frac{|T_1^+(x_j) - T_2^+(x_j)|}{6} + \frac{|I_1^+(x_j) - I_2^+(x_j)|}{6} + \frac{|F_1^+(x_j) - F_2^+(x_j)|}{6} \right) -
$$
$$
\left(\frac{|T_1^-(x_j) - T_2^-(x_j)|}{6} + \frac{|I_1^-(x_j) - I_2^-(x_j)|}{6} + \frac{|F_1^-(x_j) - F_2^-(x_j)|}{6} \right)
$$

$$
\varphi_2(x_j) = \left(\begin{array}{l} max \left\{ \frac{2+T_1^+(x_j)-I_1^+(x_j)-F_1^+(x_j)}{6}, \frac{2+T_2^+(x_j)-I_2^+(x_j)-F_2^+(x_j)}{6} \right\} \\ -min \left\{ \frac{2+T_1^+(x_j)-I_1^+(x_j)-F_1^+(x_j)}{6}, \frac{2+T_2^+(x_j)-I_2^+(x_j)-F_2^+(x_j)}{6} \right\} \end{array} \right)
$$
$$
- \left(\begin{array}{l} max \left\{ \frac{2+T_1^-(x_j)-I_1^-(x_j)-F_1^-(x_j)}{6}, \frac{2+T_2^-(x_j)-I_2^-(x_j)-F_2^-(x_j)}{6} \right\} \\ -min \left\{ \frac{2+T_1^-(x_j)-I_1^-(x_j)-F_1^-(x_j)}{6}, \frac{2+T_2^-(x_j)-I_2^-(x_j)-F_2^-(x_j)}{6} \right\} \end{array} \right)
$$

$$
\varphi_3(x_j) = \frac{|T_1^+(x_j) - T_2^+(x_j) + I_2^+(x_j) - I_1^+(x_j)|}{4} - \frac{|T_1^-(x_j) - T_2^-(x_j) + I_2^-(x_j) - I_1^-(x_j)|}{4}
$$

$$
\varphi_4(x_j) = \frac{|T_1^+(x_j) - T_2^+(x_j) + F_2^+(x_j) - F_1^+(x_j)|}{4}
$$
$$
- \frac{|T_1^-(x_j) - T_2^-(x_j) + F_2^-(x_j) - F_1^-(x_j)|}{4}
$$

Proposition3.2. The distance measure $d_\lambda(A_1, A_2)$ for $\lambda > 0$ satisfies the following properties:

(H1) $0 \le d_\lambda(A_1, A_2) \le 1$;

(H2) $d_\lambda(A_1, A_2) = 0$ if and only if $A_1 = A_2$;

(H3) $d_\lambda(A_1, A_2) = d_\lambda(A_2, A_1)$;

(H4) If $A_1 \subseteq A_2 \subseteq A_3$, A_3 is a bipolar neutrosophic in X, then $d_\lambda(A_1, A_3) \geq d_\lambda(A_1, A_2)$ and

$$d_\lambda(A_1, A_3) \geq d_\lambda(A_2, A_3).$$

Proof: It is easy to see that $d_\lambda(A_1, A_2)$ satisfies the properties $(H1) - (H3)$. Therefore, we only prove (H4).

Let $A_1 \subseteq A_2 \subseteq A_3$, then

$T_1^+(x_i) \leq T_2^+(x_i) \leq T_3^+(x_i), \; T_1^-(x_i) \geq T_2^-(x_i) \geq T_3^-(x_i) \,,$

$I_1^+(x_i) \leq I_2^+(x_i) \leq I_3^+(x_i), \;\; I_1^-(x_i) \geq I_2^-(x_i) \geq I_3^-(x_i) \,,$ and

$F_1^+(x_i) \geq F_2^+(x_i) \geq F_3^+(x_i) \,, F_1^-(x_i) \leq F_2^-(x_i) \leq F_3^-(x_i),$ for every $x_i \in X$. Then, we obtain the following relations:

$$|T_1^+(x_i) - T_2^+(x_i)| \leq |T_1^+(x_i) - T_3^+(x_i)|, |T_2^+(x_i) - T_3^+(x_i)| \leq |T_1^+(x_i) - T_3^+(x_i)|,$$

$$\left|T_1^-(x_i) - T_2^-(x_j)\right| \leq |T_1^-(x_i) - T_3^-(x_i)|, |T_2^-(x_i) - T_3^-(x_i)| \leq |T_1^-(x_i) - T_3^-(x_i)|,$$

$$|I_1^+(x_i) - I_2^+(x_i)| \leq |I_1^+(x_i) - I_3^+(x_i)| \,, |I_2^+(x_i) - I_3^+(x_i)| \leq |I_1^+(x_i) - I_3^+(x_i)|,$$

$$|I_1^-(x_i) - I_2^-(x_i)| \leq |I_1^-(x_i) - I_3^-(x_i)| \,, |I_2^-(x_i) - I_3^-(x_i)| \leq |I_1^-(x_i) - I_3^-(x_i)|,$$

$$|F_1^+(x_i) - F_2^+(x_i)| \leq |F_1^+(x_i) - F_3^+(x_i)| \,, |F_2^+(x_i) - F_3^+(x_i)| \leq |F_1^+(x_i) - F_3^+(x_i)|,$$

$$|F_1^-(x_i) - F_2^-(x_i)| \leq |F_1^-(x_i) - F_3^-(x_i)| \,, |F_2^-(x_i) - F_3^-(x_i)| \leq |F_1^-(x_i) - F_3^-(x_i)|,$$

hence,

$$|T_1^+(x_i) - T_2^+(x_i)| + |I_1^+(x_i) - I_2^+(x_i)| + |F_1^+(x_i) - F_2^+(x_i)| +$$

$$\left|T_1^-(x_i) - T_2^-(x_j)\right| + |I_1^-(x_i) - I_2^-(x_i)| + |F_1^-(x_i) - F_2^-(x_i)| \leq$$

$$|T_1^+(x_i) - T_3^+(x_i)| + |I_1^+(x_i) - I_3^+(x_i)| + |F_1^+(x_i) - F_3^+(x_i)| +$$

$$|T_1^-(x_i) - T_3^-(x_i)| + |I_1^-(x_i) - I_3^-(x_i)| + |F_1^-(x_i) - F_3^-(x_i)|,$$

$$|T_2^+(x_i) - T_3^+(x_i)| + |I_2^+(x_i) - I_3^+(x_i)| + |F_2^+(x_i) - F_3^+(x_i)| +$$

$$|T_2^-(x_i) - T_3^-(x_i)| + |I_2^-(x_i) - I_3^-(x_i)| + |F_2^-(x_i) - F_3^-(x_i)| \leq$$

$$|T_1^+(x_i) - T_3^+(x_i)| + |I_1^+(x_i) - I_3^+(x_i)| + |F_1^+(x_i) - F_3^+(x_i)| +$$

$$|T_1^-(x_i) - T_3^-(x_i)| + |I_1^-(x_i) - I_3^-(x_i)| + |F_1^-(x_i) - F_3^-(x_i)|,$$

$$\frac{2 + T_1^+(x_j) - I_1^+(x_j) - F_1^+(x_j)}{6} \le \frac{2 + T_2^+(x_j) - I_2^+(x_j) - F_2^+(x_j)}{6}$$

$$\le \frac{2 + T_3^+(x_j) - I_3^+(x_j) - F_3^+(x_j)}{6}$$

$$\frac{2 + T_1^-(x_j) - I_1^-(x_j) - F_1^-(x_j)}{6} \le \frac{2 + T_2^-(x_j) - I_2^-(x_j) - F_2^-(x_j)}{6}$$

$$\le \frac{2 + T_3^-(x_j) - I_3^-(x_j) - F_3^-(x_j)}{6}$$

$$0 \le \frac{T_2^+(x_j) - T_1^+(x_j) + I_1^+(x_j) - I_2^+(x_j)}{4} \le \frac{T_3^+(x_j) - T_1^+(x_j) + I_1^+(x_j) - I_3^+(x_j)}{4}$$

$$0 \le \frac{T_2^-(x_j) - T_1^-(x_j) + I_1^-(x_j) - I_2^-(x_j)}{4} \le \frac{T_3^-(x_j) - T_1^-(x_j) + I_1^-(x_j) - I_3^-(x_j)}{4}$$

$$0 \le \frac{T_2^+(x_j) - T_1^+(x_j) + F_1^+(x_j) - F_2^+(x_j)}{4} \le \frac{T_3^+(x_j) - T_1^+(x_j) + F_1^+(x_j) - F_3^+(x_j)}{4}$$

$$0 \le \frac{T_2^-(x_j) - T_1^-(x_j) + F_1^-(x_j) - F_2^-(x_j)}{4} \le \frac{T_3^-(x_j) - T_1^-(x_j) + F_1^-(x_j) - F_3^-(x_j)}{4}$$

$$\varphi_i^{A_1 A_2}(x_j) \le \varphi_i^{A_1 A_3}(x_j), \ \varphi_i^{A_2 A_3}(x_j) \le \varphi_i^{A_1 A_3}(x_j), \ i = 1,2,3,4 \ j = 1,2,\dots,n,$$

$$d_\lambda(A_1, A_3) \ge d_\lambda(A_1, A_2) \text{ and } d_\lambda(A_1, A_3) \ge d_\lambda(A_2, A_3) \text{ for } \lambda > 0.$$

Definition3.3. For two bipolar neutrosophic A_1 and A_2 in a universe of discourse $X = \{x_1, x_2, \dots, x_n\}$ which are denoted by $A_1 = \langle x_i, T_1^+(x_i), I_1^+(x_i), F_1^+(x_i), T_1^-(x_i), I_1^-(x_i), F_1^-(x_i) \rangle$ and $A_2 = \langle x_i, T_2^+(x_i), I_2^+(x_i), F_2^+(x_i), T_2^-(x_i), I_2^-(x_i), F_2^-(x_i) \rangle$. The bipolar neutrosophic weighted similarity measure are defined by

$$\vartheta_\lambda(A_1, A_2) = 1 - d_\lambda(A_1, A_2). \tag{2}$$

Proposition3.4. The similarity measure $\vartheta_\lambda(A_1, A_2)$ for $\lambda > 0$ satisfies the following properties;

(HD1) $0 \le \vartheta_\lambda(A_1, A_2) \le 1$;

(HD2) $\vartheta_\lambda(A_1, A_2) = 1$ if and only if $A_1 = A_2$;

(HD3) $\vartheta_\lambda(A_1, A_2) = \vartheta_\lambda(A_2, A_1)$;

(HD4) If $A_1 \subseteq A_2 \subseteq A_3$, A_3 is a bipolar neutrosophic in X, then $\vartheta_\lambda(A_1, A_2) \geq \vartheta_\lambda(A_1, A_3)$ and

$$\vartheta_\lambda(A_2, A_3) \geq \vartheta_\lambda(A_1, A_3).$$

Definition3.5. Let E be a set-to-the point mapping: E: bipolar neutrosophic $\to[0,1]$, then E is an entropy measure if it satisfies the following conditions:

(E1) $E(A_1) = 0$ (minimum) if and only if A or $A_1{}^c$ is a crisp set;

(E2) $E(A_1) = 1$(maximum) if and only if $A_1 = A_1{}^c$ $T_1^+(x_i) = F_1^+(x_i)$, $T_1^-(x_i) = F_1^-(x_i)$, $I_1^+(x_i) = I_1^-(x_i) = 0.5$ for all $x_i \in X$

(E3) $E(A_1) \leq E(A_2)$ if A_1 is less fuzzy than A_2,

 $T_1^+(x_i) \leq T_2^+(x_i)$, $F_2^+(x_i) \leq F_1^+(x_i)$, for $T_2^+(x_i) \leq F_2^+(x_i)$ and $I_1^+(x_i) = I_2^+(x_i) = 0.5$

 $T_1^-(x_i) \geq T_2^-(x_i)$, $F_2^-(x_i) \geq F_1^-(x_i)$, for $T_2^-(x_i) \geq F_2^-(x_i)$ and $I_1^-(x_i) = I_2^-(x_i) = 0.5$

or

 $T_1^+(x_i) \geq T_2^+(x_i)$, $F_2^+(x_i) \geq F_1^+(x_i)$, for $T_2^+(x_i) \geq F_2^+(x_i)$ and $I_1^+(x_i) = I_2^+(x_i) = 0.5$

 $T_1^-(x_i) \leq T_2^-(x_i)$, $F_2^-(x_i) \leq F_1^-(x_i)$, for $T_2^-(x_i) \leq F_2^-(x_i)$ and $I_1^-(x_i) = I_2^-(x_i) = 0.5$

(E4) $E(A_1) = E(A_1{}^c)$.

Remark: In some cases, we do not only think about the distance between $A_1 \ and \ A_2$, but also we need to consider the distance between A_1 and $A_2{}^c$. So we can define the index of distance for two bipolar neutrosophic A_1 and A_2 as follows.

Definition3.6. For two bipolar neutrosophic A_1 and A_2, the index of distance is defined by

$$I_\lambda(A_1, A_2) = \frac{d_\lambda(A_1, A_2)}{d_\lambda(A_1, A_2{}^c)}.$$

Proposition3.7. The index of distance $I_\lambda(A_1, A_2)$ for two bipolar neutrosophic A_1 and A_2 satisfies the following properties:

(I1) $I_\lambda(A_1, A_2) = 0$ if and only if $A_1 = A_2$;

(I2) $I_\lambda(A_1, A_2) = 0$ if and only if $d_\lambda(A_1, A_2) = d_\lambda(A_1, A_2{}^c)$;

(I3) $I_\lambda(A_1, A_2) \to +\infty, A_1 = A_2{}^c$, these means A_1 and A_2 are completely different;

(I4) $When \ A_1 = A_2 = A_2{}^c$, the entropy measure of A_1 and A_2 reaches its maximum value;

(I5) $I_\lambda(A_1, A_2) < 1$ means compare with $A_2{}^c A_1$ is more similar to A_2;

(I5) $I_\lambda(A_1, A_2) > 1$ means compare with $A_2{}^c A_1$ is less similar to A_2.

4. Practical Examples

In this section, two examples are given to demonstrate the application of the proposed distance measure.

4.1. Clustering Method Based on the Distance (Similarity) Measure of BNSs and an Example

In this subsection, we introduced a method for a MADM problem with bipolar neutrosophic information. Some of it is quoted from (Ye 2014).

Step1. By use of Equations 1 and 2, we can calculate the similarity measure degree of bipolar neutrosophic set. Then we have a similarity matrix $C = (s_{ij})_{mxm}$, where
$$s_{ij} = s_{ji} = \vartheta_\lambda(A_i, A_j) \text{ for } i, j = 1,2, \dots, m.$$

Step2. The process of building the composition matrices is repeated until it holds that
$$C \rightarrow C^2 \rightarrow C^4 \rightarrow \cdots \rightarrow C^{2^k} = C^{2^{k+1}}$$

C^{2^k} is an equivalent matrix, where
$$C^2 = C \circ C = (s'_{ij})_{mxm} = max_k\{\min(s_{ik}, s_{kj})\}_{mxm},$$
for $i, j = 1,2, \dots, m.$

Step3. For the equivalent matrix $C^{2^k} \triangleq \bar{C} = (\bar{s}_{ij})_{mxm}$, we can construct a α-cutting matrix $\bar{C}_\alpha = (\bar{s}_{ij}^\alpha)_{m \times m}$ of $\bar{C}$, where

$$\bar{s}_{ij}^\alpha = \begin{cases} 0, & \bar{s}_{ij} < \alpha; \\ 1, & \bar{s}_{ij} \geq \alpha; \end{cases}$$

for $i, j = 1,2, \dots, m$ and α is the confidence level with $\alpha \in [0,1]$.

Step4. Classify A_i by choosing different level α. Line i and k of $\bar{C}_\alpha$ are called α-congruence if $\bar{s}_{ij}^\alpha = \bar{s}_{kj}^\alpha$ for all $j = 1,2, \dots, m$. Then A_i should fall into the same category as A_k.

Example4.1.1 A car seller is going to classify four different cars of $A_m (m = 1,2,3,4)$. Every car has four evaluation factors (attributes): (1) x_1, fuel consumption; (2) x_2, coefficient of friction; (3) x_3, price; (4) x_4, comfortable degree. The characteristics of each car under the four attributes are represented by the form of bipolar neutrosophic data are as follows:

Step1. Construct the decision matrix provided by the customer as;

Table 1: Decision matrix given by customer

	x_1	x_2	x_3	x_4
A_1	$\langle 0.5,0.7,0.2,-0.7,-0.3,-0.6\rangle$	$\langle 0.6,0.4,0.5,-0.7,-0.8,-0.4\rangle$	$\langle 0.7,0.7,0.5,-0.8,-0.7,-0.6\rangle$	$\langle 0.1,0.5,0.7,-0.5,-0.2,-0.8\rangle$
A_2	$\langle 0.8,0.7,0.5,-0.7,-0.7,-0.1\rangle$	$\langle 0.7,0.6,0.8,-0.7,-0.5,-0.1\rangle$	$\langle 0.9,0.4,0.6,-0.1,-0.7,-0.5\rangle$	$\langle 0.5,0.2,0.7,-0.5,-0.1,-0.9\rangle$
A_3	$\langle 0.3,0.4,0.2,-0.6,-0.3,-0.7\rangle$	$\langle 0.2,0.2,0.2,-0.4,-0.7,-0.4\rangle$	$\langle 0.9,0.5,0.5,-0.6,-0.5,-0.2\rangle$	$\langle 0.7,0.5,0.3,-0.4,-0.2,-0.2\rangle$
A_4	$\langle 0.9,0.7,0.2,-0.8,-0.6,-0.1\rangle$	$\langle 0.3,0.5,0.3,-0.5,-0.5,-0.2\rangle$	$\langle 0.5,0.4,0.5,-0.1,-0.7,-0.2\rangle$	$\langle 0.6,0.2,0.8,-0.5,-0.5,-0.6\rangle$

Let $\lambda = 2$ choosing the weight vectors $w_j = \frac{1}{4}$ $(j = 1,2,3,4)$ and $\beta_i = \frac{1}{4}$ $(i = 1,2,3,4,)$, then we use similarity measure to classify the four different cars of $A_m(m = 1,2,3,4)$ by the bipolar neutrosophic clustering algorithms.

First, we utilize the distance measure to calculate the distance measures between each pair of bipolar neutrosophic $A_m(m = 1,2,3,4)$. The results are as follows;

$$d(A_1, A_2) = 0{,}062743, \qquad d(A_1, A_3) = 0{,}072924, \qquad d(A_1, A_4) = 0{,}069644,$$

$$d(A_2, A_3) = 0{,}061299, \qquad d(A_2, A_4) = 0{,}034264, \qquad d(A_3, A_4) = 0{,}04648,$$

Step2. So we construct the following similarity matrix:

$$C = \begin{bmatrix} 1 & 0{,}937257415 & 0{,}927075893 & 0{,}930356189 \\ 0{,}937257415 & 1 & 0{,}938700771 & 0{,}965735658 \\ 0{,}927075893 & 0{,}938700771 & 1 & 0{,}953520184 \\ 0{,}930356189 & 0{,}965735658 & 0{,}953520184 & 1 \end{bmatrix}$$

Then by Step 2

$$C^2 = \begin{bmatrix} 1 & 0{,}937257415 & 0{,}937257415 & 0{,}937257415 \\ 0{,}937257415 & 1 & 0{,}953520184 & 0{,}965735658 \\ 0{,}937257415 & 0{,}953520184 & 1 & 0{,}953520184 \\ 0{,}937257415 & 0{,}965735658 & 0{,}953520184 & 1 \end{bmatrix}$$

Due to $C^2 \nsubseteq C, C$ is not an equivalent matrix, we keep/continue calculating.

$$C^4 = \begin{bmatrix} 1 & 0{,}937257415 & 0{,}937257415 & 0{,}937257415 \\ 0{,}937257415 & 1 & 0{,}953520184 & 0{,}965735658 \\ 0{,}937257415 & 0{,}953520184 & 1 & 0{,}953520184 \\ 0{,}937257415 & 0{,}965735658 & 0{,}953520184 & 1 \end{bmatrix}$$

$C^2 = C^4$. That is C^2 is an equivalent matrix, denoted by $\bar{C}$.

Step3. Finally, choosing different confidence level α, we can construct a α-cutting matrix $\bar{C}_\alpha$;

$\bar{C}_\alpha = (\bar{s}_{ij}^\alpha)_{m \times m}$ of $\bar{C}$, where

$$\bar{s}_{ij}^{\alpha} = \begin{cases} 0, & \bar{s}_{ij} < \alpha; \\ 1, & \bar{s}_{ij} \geq \alpha; \end{cases}$$

(1) Let $0 \leq \alpha \leq 0{,}937257415$, $\bar{C}_{\alpha} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ $A_m (m = 1{,}2{,}3{,}4)$ can be

divided into one category $\{A_1, A_2, A_3, A_4\}$.

(2) Let $0{,}937257415 < \alpha \leq 0{,}953520184$, $\bar{C}_{\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ Then the cars

$A_m (m = 1{,}2{,}3{,}4)$ can be divided into two categories $\{A_1\}, \{A_2, A_3, A_4\}$.

(3) Let $0{,}953520184 < \alpha \leq 0{,}965735658$, $\bar{C}_{\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ Then the cars

$A_m (m = 1{,}2{,}3{,}4)$ can be divided into three categories $\{A_1\}, \{A_2, A_4\}, \{A_3\}$.

(4) Let $0{,}965735658 < \alpha \leq 1$, $\bar{C}_{\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Step4. Then the cars $A_m (m = 1{,}2{,}3{,}4)$ can be divided into four categories $\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}$.

4.2. Multi-criteria Decision Making

Example 4.2.1 A manufacturing company which wants to select the global supplier according to the core competencies of suppliers. Now suppose that there are a set of four $A = \{A_1, A_2, A_3, A_4\}$ whose core competence are evaluated by means of the following four criteria $\{x_1, x_2, x_3, x_4\}$.

x_1: the level of technology innovation, x_2: the control ability of flow, x_3: the ability of management, x_4: the level of service. Then, the weight vector for the four criteria is $w = (0{,}4, 0{,}1, 0{,}3, 0{,}2)$. When the four possible alternatives with respect to the above four criteria are evaluated by the similar method from the expert, we can obtain the following bipolar neutsophic decision matrix A:

A

$$= \begin{bmatrix} \langle 0.3,0.4,0.2,-0.6,-0.3,-0.7 \rangle & \langle 0.8,0.6,0.3,-0.2,-0.6,-0.5 \rangle & \langle 0.4,0.5,0.2,-0.5,-0.1,-0.7 \rangle & \langle 0.8,0.4,0.2,-0.6,-0.3,-0.7 \rangle \\ \langle 0.3,0.6,0.9,-0.5,-0.3,-0.5 \rangle & \langle 0.3,0.3,0.1,-0.5,-0.1,-0.1 \rangle & \langle 0.6,0.6,0.3,-0.6,-0.1,-0.7 \rangle & \langle 0.3,0.4,0.2,-0.4,-0.2,-0.7 \rangle \\ \langle 0.2,0.8,0.2,-0.6,-0.8,-0.7 \rangle & \langle 0.9,0.2,0.2,-0.6,-0.6,-0.3 \rangle & \langle 0.8,0.8,0.4,-0.9,-0.3,-0.8 \rangle & \langle 0.6,0.4,0.2,-0.5,-0.3,-0.7 \rangle \\ \langle 0.6,0.4,0.5,-0.2,-0.3,-0.1 \rangle & \langle 0.7,0.5,0.1,-0.9,-0.3,-0.7 \rangle & \langle 0.3,0.9,0.5,-0.1,-0.3,-0.5 \rangle & \langle 0.3,0.4,0.9,-0.6,-0.3,-0.1 \rangle \end{bmatrix}$$

by applying Definition 3.1 the distance between an alternative A_i ($i = 1,2,3,4$) and the alternative

$$A^* = \langle max\{T_{ij}^+\}, min\{I_{ij}^+\}, min\{F_{ij}^+\}, min\{T_{ij}^-\}, max\{I_{ij}^-\}, max\{F_{ij}^-\}\rangle (j = 1,2 \dots n).$$

$A^* = \langle 0.6, 0.4, 0.2, -0.6, -0.3, -0.1\rangle\ \langle 0.9, 0.2, 0.1, -0.9, -0.1, -0.1\rangle\ \langle 0.8, 0.5, 0.2, -0.9, -0.1, -0.5\rangle$
$\langle 0.8, 0.4, 0.2, -0.6, -0.2, -0.1\rangle$

are as follows:

$$d(A_1, A^*) = 0{,}03308, \qquad d(A_2, A^*) = 0{,}0901,$$

$$d(A_3, A^*) = 0{,}03383, \qquad d(A_4, A^*) = 0{,}0688$$

with $\lambda = 2$ and $\beta_i = {}^1\!/_4$ ($i = 1,2,3,4$). $A_1 < A_3 < A_4 < A_2$.This implies that the ranking order of the four suppliers is $A_1, A_3, A_4\ and\ A_2$. Therefore, the best supplier is A_1.

5. Comparison Analysis and Discussion

In order to verify the feasibility and effectiveness of the proposed decision-making approach, a comparison analysis with single-valued neutrosophic decision method, used by (Huang 2016), is given, based on the same illustrative example.

Clearly, the ranking order results are consistent with the result obtained in (Huang 2016); however, the best alternative is the same as A_1, because the ranking principle is different, these two methods produced the same best alternative whiles/whereas the bad ones differ from each other.

As mentioned above, the bipolar single-valued neutrosophic information is a generalization of single-valued neutrosophic information, intuitionistic fuzzy information which is a further generalization of fuzzy information. On the one hand, a SVNS is an instance of a neutrosophic set, which gives us an additional possibility to represent uncertain, imprecise, incomplete, and inconsistent information that exist in the real world. On the other hand, the clustering analysis under a bipolar single-valued neutrosophic environment is suitable for capturing imprecise, uncertain, and inconsistent information in clustering the data. Thus, the clustering algorithm based on the similarity measures of BSVNSs can not only cluster the bipolar single-valued neutrosophic information but also can cluster the single-valued neutrosophic information, intuitionistic fuzzy information and the fuzzy information. Obviously, the proposed bipolar single-valued neutrosophic clustering algorithm is the extension of fuzzy clustering algorithm, single-valued neutrosophic clustering algorithm and intuitionistic fuzzy clustering algorithm. Therefore, compared with the intuitionistic fuzzy clustering algorithm, single-valued neutrosophic clustering algorithm and the fuzzy clustering algorithm, the bipolar single-valued neutrosophic clustering algorithm is more

general. Furthermore, when we encounter some situations that are represented by indeterminate information and inconsistent information, the bipolar single-valued neutrosophic clustering algorithm can demonstrate its great superiority in clustering those bipolar single-valued neutrosophic data.

6. Conclusion

BNSs can be applied in addressing problems with uncertain, imprecise, incomplete and inconsistent information existing in real scientific and engineering applications. Based on related research achievements in BNSs, we defined a new distance measure. It is a generalization of the existing distance measures defined in (Huang 2016). Then, we also defined a new similarity measure, an entropy measure, and an index of distance under the single-valued neutrosophic environment. Two illustrative examples demonstrated the application of the proposed clustering analysis method and decision-making method.

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SECTION THREE

OTHER PAPERS

Chapter Eleven

Dice Vector Similarity Measure of Intuitionistic Trapezoidal Fuzzy Multi-Numbers and Its Application in Architecture

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Abstract

This paper presents a study on the development of a intuitionistic fuzzy multi-criteria decision-making model for the evaluation of end products of the architectural of material, design and application. The main aim of this study is to present a novel method based on multi-criteria decision making Intuitionistic trapezoidal fuzzy multi-number. Therefore, Dice vector similarity measure is defined to develop the Intuitionistic Trapezoidal Fuzzy Multi-Numbers, the application of architecture are presented.

Keywords: Intuitionistic Trapezoidal Fuzzy Multi-Numbers, Dice vector similarity measure, multi-criteria decision making, architecture.

1.Introduction

In parallel with changing and developing technology, architecture and interior architecture areas have a rapidly rising graphic within the context of material, design and application. Within this context, for the purpose of producing design alternatives in a shorter time and introducing more preferences to the user, new expression procedures, in other words digital environments have been initiated to be used. Prior to designing the space, the interaction and communication between the space and its user should be solved, the person is continuously in communication with the space where he is. Therefore, the method proposed in this study will help decision-making in the most appropriate to space. In 1965, Zadeh [35] proposed fuzzy sets to handle imperfect, vague, uncertain and imprecise information as a fuzzy subset of the classical universe set A. Soon after the definition of fuzzy set, the set has been successfully applied in engineering, game theory, multi-agent systems, control systems, decision-making and so on. In the fuzzy sets, an element in a universe has a membership value in [0, 1]; however, the membership value is inadequate for providing complete information in some problems as there are situations where each element has different membership values. For this reason, a different generalization of

fuzzy sets, namely multi-fuzzy sets, has been introduced. Yager [38] first proposed multi-fuzzy sets as a generalization of multisets and fuzzy sets. An element of a multi-fuzzy set may possess more-than-one membership value in [0, 1] (or there may be repeated occurrences of an element). Some Works on the multi-sets have been undertaken by Sebastian and Ramakrishnan [19], Syropoulos [20, 21], Maturo [8], Miyamoto [6, 7] and so on. Recently, research on fuzzy numbers, with the universe of discourse as the real line, has studied. For example, Thowhida and Ahmad [25] introduced some arithmetic operations on fuzzy numbers with linear membership functions. Chakrabort and Guha [4] developed some arithmetic operations on generalized fuzzy numbers by using extension principle. Alim et al. [1] developed a formula for the elementary operations on L-R fuzzy number. Roseline and Amirtharaj [15] proposed a method of ranking of generalized trapezoidal fuzzy numbers and developed generalized fuzzy Hungarian method to find the initial solution of generalized trapezoidal fuzzy transportation problems. Also, same authors in [16] introduced a method of ranking of generalized trapezoidal fuzzy numbers based on rank, perimeter, mode, divergence and spread. Meng et al. [9] solved a multiple attribute decision-making problem with attribute values within triangular fuzzy numbers based on the mean area measurement method. Surapati and Biswas [18] examined a multi-objective assignment problem with imprecise costs, time and ineffectiveness instead of its precise information in fuzzy numbers. Wang [29] studied preference relation with membership function representing preference degree to compare two fuzzy numbers, and relative preference relation is constructed on the fuzzy preference relation to rank a set of fuzzy numbers. Sinova et al. [23] proposed a characterization of the distribution of some random elements by extending the moment-generating function in fuzzy numbers. Riera and Torrens [14] developed a method on discrete fuzzy numbers to model complete and incomplete qualitative information. Different studies for fuzzy numbers in the recent literature have been researched. For example; on in disaster responses, emergency decision makers [17], on existence, uniqueness, calculus and properties of triangular approximations of fuzzy numbers [2], on two-dimensional discrete fuzzy numbers [30], on ranking generalized exponential trapezoidal fuzzy numbers [12], on probabilistic approach to the arithmetics of fuzzy numbers [24], on matrix games with pay-offs of triangular fuzzy numbers [3], on defuzzification of generalized fuzzy numbers [11], on fuzzy linguistic model based on discrete fuzzy numbers [13], on possibilistic characterization function of fuzzy number [22] and so on.

2. Preliminary

Let us start with some basic concepts related to fuzzy set, multi-fuzzy set, intuitionistic fuzzy set[37], intuitionistic fuzzy multiset and intuitionistic fuzzy numbers.

Definition 2.1[35] Let X be a non-empty set. A fuzzy set F on X is defined as:

$$F = \{\langle x, \mu_F(x) \rangle : x \in X\} \text{ where } \mu_F : X \to [0,1] \text{ for } x \in X.$$

Definition 2.2[34] t-norms are associative, monotonic and commutative two valued functions t that map from $[0,1] \times [0,1]$ into $[0,1]$. These properties are formulated with the following conditions:

1. $t(0,0) = 0$ and $t(\mu_{x_1},(x),1) = t(1, \mu_{x_1}(x)) = \mu_{x_1}(x)$

2. If $\mu_{x_1}(x) \le \mu_{x_3}(x)$ and $\mu_{x_2}(x) \le \mu_{x_4}(x)$, then $t(\mu_{x_1}(x), \mu_{x_2}(x)) \le t(\mu_{x_3}(x), \mu_{x_4}(x))$,

3. $t(\mu_{x_1}(x), \mu_{x_2}(x)) = t(\mu_{x_2}(x), \mu_{x_1}(x))$,

4. $t(\mu_{x_1}(x), t(\mu_{x_2}(x), \mu_{x_3}(x))) = t(t(\mu_{x_1}(x), \mu_{x_2})(x), \mu_{x_3}(x))$

Definition 2.3[34] s-norm are associative, monotonic and commutative two placed functions s which map from $[0,1] \times [0,1]$ into $[0,1]$. These properties are formulated with the following conditions:

1. $s(1,1) = 1$ and $s(\mu_{x_1}(x),0) = s(0, \mu_{x_1}(x)) = \mu_{x_1}(x)$,

2. If $\mu_{x_1}(x) \le \mu_{x_3}(x)$ and $\mu_{x_2}(x) \le \mu_{x_4}(x)$, then $s(\mu_{x_1}(x), \mu_{x_2}(x)) \le s(\mu_{x_3}(x), \mu_{x_4}(x))$,

3. $s(\mu_{x_1}(x), \mu_{x_2}(x)) = s(\mu_{x_2}(x), \mu_{x_1}(x))$,

4. $s(\mu_{x_1}(x), s(\mu_{x_2}(x), \mu_{x_3}(x))) = s(s(\mu_{x_1}(x), \mu_{x_2})(x), \mu_{x_3}(x))$.

t-norm and t-conorm is related in a sense of logical duality. Typical dual pairs of non-parametrized t-norm and t-conorm are compiled below:

1. Drastic product: $t_w(\mu_{x_1}(x), \mu_{x_2}(x)) = \begin{cases} \min\{\mu_{x_1}(x), \mu_{x_2}(x)\}, \max\{\mu_{x_1}(x), \mu_{x_2}(x)\} = 1 \\ \qquad 0 \qquad\qquad\qquad otherwise \end{cases}$

2. Drastic sum: $s_w(\mu_{x_1}(x), \mu_{x_2}(x)) = \begin{cases} \max\{\mu_{x_1}(x), \mu_{x_2}(x)\}, \min\{\mu_{x_1}(x), \mu_{x_2}(x)\} = 0 \\ \qquad 1 \qquad\qquad\qquad otherwise \end{cases}$

3. Bounded product:

$t_1(\mu_{x_1}(x), \mu_{x_2}(x)) = \max\{0, \mu_{x_1}(x) + \mu_{x_2}(x) - 1\}$

4. Bounded sum:

$$s_1(\mu_{x_1}(\mathrm{x}), \mu_{x_2}(\mathrm{x})) = \min\left\{1, \mu_{x_1}(\mathrm{x}) + \mu_{x_2}(\mathrm{x})\right\}$$

5. Einstein product:

$$t_{1.5}(\mu_{x_1}(\mathrm{x}), \mu_{x_2}(\mathrm{x})) = \frac{\mu_{x_1}(\mathrm{x}).\mu_{x_2}(\mathrm{x})}{2 - [\mu_{x_1}(\mathrm{x}) + \mu_{x_2}(\mathrm{x}) - \mu_{x_1}(\mathrm{x}).\mu_{x_2}(\mathrm{x})]}$$

6. Einstein sum:

$$s_{1.5}(\mu_{x_1}(\mathrm{x}), \mu_{x_2}(\mathrm{x})) = \frac{\mu_{x_1}(\mathrm{x}) + \mu_{x_2}(\mathrm{x})}{1 + \mu_{x_1}(\mathrm{x}).\mu_{x_2}(\mathrm{x})}$$

7. Algebraic product:

$$t_2(\mu_{x_1}(\mathrm{x}), \mu_{x_2}(\mathrm{x})) = \mu_{x_1}(\mathrm{x}).\mu_{x_2}(\mathrm{x})$$

8. Algebraic sum:

$$s_2(\mu_{x_1}(\mathrm{x}), \mu_{x_2}(\mathrm{x})) = \mu_{x_1}(\mathrm{x}) + \mu_{x_2}(\mathrm{x}) - \mu_{x_1}(\mathrm{x}).\mu_{x_2}(\mathrm{x})$$

9. Hamacher product:

$$t_{2.5}(\mu_{x_1}(\mathrm{x}), \mu_{x_2}(\mathrm{x})) = \frac{\mu_{x_1}(\mathrm{x}).\mu_{x_2}(\mathrm{x})}{\mu_{x_1}(\mathrm{x}) + \mu_{x_2}(\mathrm{x}) - \mu_{x_1}(\mathrm{x}).\mu_{x_2}(\mathrm{x})}$$

10. Hamacher Sum:

$$s_{2.5}(\mu_{x_1}(\mathrm{x}), \mu_{x_2}(\mathrm{x})) = \frac{\mu_{x_1}(\mathrm{x}) + \mu_{x_2}(\mathrm{x}) - 2.\mu_{x_1}(\mathrm{x}).\mu_{x_2}(\mathrm{x})}{1 - \mu_{x_1}(\mathrm{x}).\mu_{x_2}(\mathrm{x})}$$

11. Minimum:

$$t_3(\mu_{x_1}(\mathrm{x}), \mu_{x_2}(\mathrm{x})) = \min\left\{\mu_{x_1}(\mathrm{x}), \mu_{x_2}(\mathrm{x})\right\}$$

12. Maximum:

$$s_3(\mu_{x_1}(\mathrm{x}), \mu_{x_2}(\mathrm{x})) = \max\left\{\mu_{x_1}(\mathrm{x}), \mu_{x_2}(\mathrm{x})\right\}$$

Definition 2.4 [19] Let X be a non-empty set. A multi-fuzzy set G on X is defined as $G = \left\{ \left\langle x, \mu_G^1(x), \mu_G^2(x), ..., \mu_G^i(x), ... \right\rangle : x \in X \right\}$ where $\mu_G^i : X \to [0,1]$ for all $i \in \{1, 2, ..., p\}$ and $x \in X$

Definition 2.5[32] Let $\eta_A^i, v_A^i \in [0,1]$ $(i \in \{1, 2, ..., p\})$ and $a, b, c, d \in \mathbb{R}$ such that $a \le b \le c \le d$. Then, an intuitionistic trapezoidal fuzzy multi-number (ITFM number) $\tilde{a} = \left\langle [a, b, c, d]; \left(\eta_A^1, \eta_A^2, ..., \eta_A^p \right), \left(v_A^1, v_A^2, ..., v_A^p \right) \right\rangle$ is a special intuitionistic fuzzy multi-set on the real number set $\mathbb{R}$, whose membership functions and non-membership functions are defined as follows, respectively:

$$\mu_A^i(x) = \begin{cases} (x-a)\eta_A^i / (b-a) & a \le x \le b \\ \eta_A^i, & b \le x \le c \\ (d-x)\eta_A^i / (d-c) & c \le x \le d \\ 0, & otherwise \end{cases}$$

$$v_A^i(x) = \begin{cases} \dfrac{(b-x) + v_A^i(x-a)}{(b-a)}, & a \le x \le b \\ v_A^i, & b \le x \le c \\ \dfrac{(x-c) + v_A^i(d-x)}{(d-c)} & c \le x \le d \\ 1, & otherwise. \end{cases}$$

Note that the set of all ITFM-number on $\mathbb{R}$ will be denoted by Ω.

Definition 2.6 [32] Let $A = \left\langle [a_1, b_1, c_1, d_1]; \left(\eta_A^1, \eta_A^2, ..., \eta_A^p \right), \left(v_A^1, v_A^2, ..., v_A^p \right) \right\rangle$, $B = \left\langle [a_2, b_2, c_2, d_2]; \left(\eta_A^1, \eta_A^2, ..., \eta_A^p \right), \left(v_A^1, v_A^2, ..., v_A^p \right) \right\rangle \in \Omega$ and $\gamma \ne 0$ be any real number. Then,

1. $A + B = \left\langle [a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2]; \left(s(\eta_A^1, \eta_B^1), s(\eta_A^2, \eta_B^2), ..., s(\eta_A^P, \eta_B^P) \right), \left(t(v_A^1, v_B^1), t(v_A^2, v_B^2), ..., t(v_A^P, v_B^P) \right) \right\rangle$.
2. $A - B = \left\langle [a_1 - a_2, b_1 - b_2, c_1 - c_2, d_1 - d_2]; \left(s(\eta_A^1, \eta_B^1), s(\eta_A^2, \eta_B^2), ..., s(\eta_A^P, \eta_B^P) \right), \left(t(v_A^1, v_B^1), t(v_A^2, v_B^2), ..., t(v_A^P, v_B^P) \right) \right\rangle$.

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3.

$$A.B = \begin{cases} \begin{aligned} &\langle [a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2]; \\ &\left(t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \right), \left(s(v_A^1, v_B^1), s(v_A^2, v_B^2), ..., s(v_A^P, v_B^P) \right) \rangle \end{aligned} & (d_1 > 0, d_2 > 0) \\[2mm] \begin{aligned} &\langle [a_1 d_2, b_1 c_2, c_1 b_2, d_1 a_2]; \\ &\left(t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \right), \left(s(v_A^1, v_B^1), s(v_A^2, v_B^2), ..., s(v_A^P, v_B^P) \right) \rangle \end{aligned} & (d_1 < 0, d_2 > 0) \\[2mm] \begin{aligned} &\langle [d_1 d_2, c_1 c_2, b_1 b_2, a_1 a_2]; \\ &\left(t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \right), \left(s(v_A^1, v_B^1), s(v_A^2, v_B^2), ..., s(v_A^P, v_B^P) \right) \rangle \end{aligned} & (d_1 < 0, d_2 < 0) \end{cases}$$

4.

$$A/B = \begin{cases} \begin{aligned} &\langle [a_1/d_2, b_1/c_2, c_1/b_2, d_1/a_2]; \\ &\left(t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \right), \left(s(v_A^1, v_B^1), s(v_A^2, v_B^2), ..., s(v_A^P, v_B^P) \right) \rangle \end{aligned} & (d_1 > 0, d_2 > 0) \\[2mm] \begin{aligned} &\langle [d_1/d_2, c_1/c_2, b_1/b_2, a_1/a_2]; \\ &\left(t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \right), \left(s(v_A^1, v_B^1), s(v_A^2, v_B^2), ..., s(v_A^P, v_B^P) \right) \rangle \end{aligned} & (d_1 < 0, d_2 > 0) \\[2mm] \begin{aligned} &\langle [d_1/a_2, c_1/b_2, b_1/c_2, a_1/d_2]; \\ &\left(t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \right), \left(s(v_A^1, v_B^1), s(v_A^2, v_B^2), ..., s(v_A^P, v_B^P) \right) \rangle \end{aligned} & (d_1 < 0, d_2 < 0) \end{cases}$$

5.

$$\gamma A = \langle [\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1]; \left(1-(1-\eta_A^1)^\gamma, 1-(1-\eta_A^2)^\gamma, ..., 1-(1-\eta_A^p)^\gamma \right), \left((v_A^1)^\gamma, (v_A^2)^\gamma, ..., (v_A^P)^\gamma \right) \rangle \; (\gamma \geq 0)$$

6.

$$A^\gamma = \langle [a_1^\gamma, b_1^\gamma, c_1^\gamma, d_1^\gamma]; \left((\eta_A^1)^\gamma, (\eta_A^2)^\gamma, ..., (\eta_A^P)^\gamma \right), \left((1-(1-v_A^1)^\gamma, 1-(1-v_A^2)^\gamma, ..., 1-(1-v_A^p)^\gamma) \right) \rangle \; (\gamma \geq 0)$$

Definition 2.7 [32] Let $A = \langle [a_1, b_1, c_1, d_1]; (\eta_A^1, \eta_A^2, ..., \eta_A^p), (v_A^1, v_A^2, ..., v_A^p) \rangle$,Then, normalized ITFM-number of A is given by

$$\bar{A} = \left\langle \left[\frac{a_1}{a_1+b_1+c_1+d_1}, \frac{b_1}{a_1+b_1+c_1+d_1}, \frac{c_1}{a_1+b_1+c_1+d_1}, \frac{d_1}{a_1+b_1+c_1+d_1} \right]; (\eta_A^1, \eta_A^2, ..., \eta_A^p), (v_A^1, v_A^2, ..., v_A^p) \right\rangle$$

Example 2.8 Assume that $A = \langle [3,5,6,7]; (0.03, 0.02, ..., 0.08), (0,001, 0,003, ..., 0,004) \rangle \in \Omega$.

Then normalized ITFM-number of A can be written as

$$\bar{A} = \left\langle \left[\frac{3}{21}, \frac{5}{21}, \frac{6}{21}, \frac{7}{21} \right]; (0.03, 0.02, ..., 0.08), (0,001, 0,003, ..., 0,004) \right\rangle \in \Omega.$$

Definition 2.9[36] Let $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ be the two vectors of length n where all the coordinates are positive. Then the Dice similarity measure between two vectors (Dice 1945) is defined as follows:

$$D(X, Y) = \frac{2X.Y}{[\![X]\!]_2^2 + [\![Y]\!]_2^2} = \frac{2\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2}$$

where $X.Y = \sum_{i=1}^{n} x_i y_i$ is the inner product of the vectors X and Y and $[\![X]\!]_2 = \sqrt{\sum_{i=1}^{n} x^2}$ and $[\![Y]\!]_2 = \sqrt{\sum_{i=1}^{n} y^2}$ are the Euclidean norms of X and Y (also called the L_2 norms). However, it is undefined if $x_i = y_i = 0$ for $i = 1, 2, ..., n$. In this case, let the measure value be zero when $x_i = y_i = 0$ for $i = 1, 2, ..., n$.

The Dice similarity measures at isfies the following properties

(P1) $0 \leq D(X, Y) \leq 1$;

(P2) $D(X, Y) = D(Y, X)$

  (P3) $D(X, Y) = 1$ if and only if $X = Y$,i.e. $x_i = y_i$, for i=1,2,...,n.

The Dice similarity measure in vector space can be extended to the following expected Dice similarity measure for intuitionistic trapezoidal fuzzy numbers.

3. Dice Vector Similarity Measure Based on Multi-Criteria Decision Making with ITFMN

Definition 3.1 Let $A = \langle [a_1, b_1, c_1, d_1]; (\eta_A^1, \eta_A^2, ..., \eta_A^p), (v_A^1, v_A^2, ..., v_A^p) \rangle$,
$B = \langle [a_2, b_2, c_2, d_2]; (\eta_A^1, \eta_A^2, ..., \eta_A^p), (v_A^1, v_A^2, ..., v_A^p) \rangle$ be two ITFMNs in the set of real

numbers $\mathbb{R}$. Then; Dice similarity measure between ITFMN A and B denoted $D(\bar{A},\bar{B})$ is defined as;

$$D(A,B) = \frac{1}{n}\sum\nolimits_{j=1}^{n}\left(\frac{\dfrac{1}{1+d(A,B)}\cdot}{\dfrac{2.(\eta_A^1(x_j).\eta_B^1(x_j)+...+\eta_A^P(x_j).\eta_B^P(x_j)).(v_A^1(x_j).v_B^1(x_j)+...+v_A^P(x_j).v_B^P(x_j))}{\left((\eta_A^1)^2(x_j)+...+(\eta_A^P)^2(x_j)+(\eta_B^P)^2(x_j)\right).\left((v_A^1)^2(x_j)+...+(v_A^P)^2(x_j)+(v_B^P)^2(x_j)\right)}}\right)$$

$$d(A,B) = \left|P(A)-P(B)\right|$$

$$P(A) = \frac{a_1+2a_2+2a_3+a_4}{6}, \quad P(B) = \frac{b_1+2b_2+2b_3+b_4}{6}$$

Example3.2 Let $A = \left\langle[1,3,5,7];(0.3,0.2,0.4,0.6),(0.2,0.1,0.4,0.5)\right\rangle$,

$B = \left\langle[2,6,7,8];(0.1,0.5,0.7,0.8),(0.3,0.6,0.7,0.5)\right\rangle$ be two ITFMNs in the set of real numbers $\mathbb{R}$. Then; Dice similarity measure between ITFMN A and B

$$d(A,B) = \left|P(A)-P(B)\right| = \left|3-6\right| = 3$$

$$P(A) = \frac{1+6+10+7}{6} = \frac{18}{6} = 3$$

$$P(B) = \frac{2+10+14+8}{6} = \frac{36}{6} = 6$$

$$D(A,B) =$$

$$\frac{1}{4}\sum\nolimits_{j=1}^{n}\left(\frac{1}{1+3}\cdot\frac{2.(0,3.0,1+0,2.0,5+0,4.0,7+0,6.0,8).(0,2.0,3+0,1.0,6+0,4.0,7+0,5.0,5)}{(0,3)^2+(0,2)^2+(0,4)^2+(0,6)^2+...+(0,3)^2+(0,6)^2+(0,7)^2+(0,5)^2}\right)$$

$$= \frac{1}{4}\cdot\frac{1}{1+3}\cdot\frac{2.(0,5785)}{3,69} = \frac{1,157}{4.4.3,69} = \frac{1,157}{59,04} \cong 0.0195$$

Proposition3.3 Let $D(A,B)$ be a Dice similarity measure between normalized ITFMN's A and B. Then we have,

i. $0 \le D(\bar{A},\bar{B}) \le 1$

ii. $D(\bar{A},\bar{B}) = D(\bar{B},\bar{A})$

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iii. $D(\overline{A}, \overline{B}) = 1$ *for* $\overline{A} = \overline{B}$

Definition 3.4 Let $A = \left\langle [a_1, b_1, c_1, d_1]; \left(\eta_A^1, \eta_A^2, ..., \eta_A^p\right), \left(v_A^1, v_A^2, ..., v_A^p\right) \right\rangle$,

$B = \left\langle [a_2, b_2, c_2, d_2]; \left(\eta_A^1, \eta_A^2, ..., \eta_A^p\right), \left(v_A^1, v_A^2, ..., v_A^p\right) \right\rangle$ two ITFMNs in the set of real numbers

$\mathbb{R}$ and $w_i \in [0,1]$ be the weight of each element x_j for $j = (1, 2, ..., n)$ such that $\sum_{i=1}^{n} w_j = 1$.

Then; Dice similarity measure between normalized ITFMN A and B denoted $D_w(\overline{A}, \overline{B})$ is defined as;

$$D_w(\overline{A}, \overline{B}) = \frac{1}{n} \sum_{j=1}^{n} \left(\frac{\dfrac{1}{1 + d(\overline{A}, \overline{B})} \cdot}{\dfrac{2.w_j.(\eta_A^1(x_j).\eta_B^1(x_j) + ... + \eta_A^p(x_j).\eta_B^p(x_j)).(v_A^1(x_j).v_B^1(x_j) + ... + v_A^p(x_j).v_B^p(x_j))}{\left((\eta_A^1)^2(x_j) + ... + (\eta_A^p)^2(x_j) + (\eta_B^p)^2(x_j)\right).\left((v_A^1)^2(x_j) + ... + (v_A^p)^2(x_j) + (v_B^P)^2(x_j)\right)}} \right)$$

$$d(\overline{A}, \overline{B}) = \left| P(\overline{A}) - P(\overline{B}) \right|$$

$$P(\overline{A}) = \frac{a_1 + 2a_2 + 2a_3 + a_4}{6}, \quad P(\overline{B}) = \frac{b_1 + 2b_2 + 2b_3 + b_4}{6}$$

Example 3.5 Let $\overline{A} = \left\langle [0.2, 0.3, 0.5, 0.6]; (0.1, 0.4, 0.6, 0.7), (0.2, 0.5, 0.6, 0.8) \right\rangle$,

$\overline{B} = \left\langle [0.1, 0.2, 0.4, 0.5]; (0.2, 0.3, 0.4, 0.7), (0.1, 0.2, 0.3, 0.4) \right\rangle$ be two normalized ITFMNs in

the set of real numbers $\mathbb{R}$ and w_i be the weight of each element x_i for $i = (1, 2)$

$w_1 = 0.3, w_2 = 0.7$ such that $\sum_{i=1}^{n} w_i = 1$. Then; Dice similarity measure between normalized

ITFMN $\overline{A}$ and $\overline{B}$ is;

$$d(\overline{A}, \overline{B}) = \left| P(\overline{A}) - P(\overline{B}) \right| = |0.4 - 0.3| = 0.1$$

$$P(\overline{A}) = \frac{0.2 + 0.6 + 0.10 + 0.6}{6} = \frac{0.24}{6} = 0.4$$

$$P(B) = \frac{0.1 + 0.4 + 0.8 + 0.5}{6} = \frac{0.18}{6} = 0.3$$

$$D_w(A,B) = \frac{1}{(1+0.1)} \cdot \frac{2.(0,3).(0,1.0,2+0,4.0,3+0,6.0,4+0,7.0,7).(0,2.0,1+0,5.0,2+0,6.0,3+0,8.0,4)}{(0,1)^2 + (0,4)^2 + (0,6)^2 + (0,7)^2 + ... + (0,1)^2 + (0,2)^2 + (0,3)^2 + (0,4)^2}$$

$$+ \cdot \frac{1}{(1+0.1)} \cdot \frac{2.(0,7).(0,1.0,2+0,4.0,3+0,6.0,4+0,7.0,7).(0,2.0,1+0,5.0,2+0,6.0,3+0,8.0,4)}{(0,1)^2 + (0,4)^2 + (0,6)^2 + (0,7)^2 + ... + (0,1)^2 + (0,2)^2 + (0,3)^2 + (0,4)^2}$$

$$= \frac{1}{1,1} \cdot \frac{2.(0,3).(1,19)}{3,39} + \frac{1}{1,1} \cdot \frac{2.(0,7).(0,5394)}{3,39}$$

$$= 0,3938$$

Proposition3.6 Let $D_w(\overline{A},\overline{B})$ be a weighted Dice similarity measure between normalized ITFMN's A and B, $w_j \in [0,1]$ be the weight of each element x_i such that $\sum_{j=1}^n w_j = 1$ Then weighted Dice vector similarity measure between ITFMN's A and B;

i. $0 \le D_w(\overline{A},\overline{B}) \le 1$

ii. $D_w(\overline{A},\overline{B}) = D_w(\overline{B},\overline{A})$

iii. $D_w(\overline{A},\overline{B}) = 1$ *for* $\overline{A} = \overline{B}$ i.e. $(\eta_A^1 = \eta_B^1, \eta_A^2 = \eta_B^2,...,\eta_A^p = \eta_B^p)$

Proof 3.7

i. It is clear from Definition 3.3

ii.

$$D_w(\overline{A},\overline{B}) = \frac{1}{n}\sum_{j=1}^n \left(\frac{\frac{1}{1+d(\overline{A},\overline{B})} \cdot}{2.w_j.(\eta_A^1(x_j).\eta_B^1(x_j)+...+\eta_A^p(x_j).\eta_B^p(x_j)).(v_A^1(x_j).v_B^1(x_j)+...+v_A^p(x_j).v_B^p(x_j))}{\left((\eta_A^1)^2(x_j)+...+(\eta_A^p)^2(x_j)+(\eta_B^p)^2(x_j)\right).\left((v_A^1)^2(x_j)+...+(v_A^p)^2(x_j)+(v_B^p)^2(x_j)\right)} \right)$$

$$= \frac{1}{n} \sum_{j=1}^{n} \left(\frac{\frac{1}{1+d(\overline{B},A)} \cdot }{\frac{2.w_j.(\eta_B^1(x_j).\eta_A^1(x_j)+...+\eta_B^p(x_j).\eta_A^p(x_j)).(v_B^1(x_j).v_A^1(x_j)+...+v_B^p(x_j).v_A^p(x_j))}{\left((\eta_B^1)^2(x_j)+(\eta_A^1)^2(x_j)+...+(\eta_B^P)^2(x_j)+(\eta_A^P)^2(x_j)\right).\left((v_B^1)^2(x_j)+...+(v_B^P)^2(x_j)+(v_A^P)^2(x_j)\right)}} \right)$$

$$= D_w(\overline{B},\overline{A})$$

iii.

$$D_w(\overline{A},\overline{B}) =$$

$$\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\frac{1}{1+d(\overline{A},\overline{B})} \cdot }{\frac{2.w_j.(\eta_A^1(x_j).\eta_B^1(x_j)+...+\eta_A^p(x_j).\eta_B^p(x_j)).(v_A^1(x_j).v_B^1(x_j)+...+v_A^p(x_j).v_B^p(x_j))}{\left((\eta_A^1)^2(x_j)+(\eta_B^1)^2(x_j)+...+(\eta_A^P)^2(x_j)+(\eta_B^P)^2(x_j)\right).\left((v_A^1)^2(x_j)+...+(v_A^P)^2(x_j)+(v_B^P)^2(x_j)\right)}} \right)$$

$$= \sum_{j=1}^{n} \left(\frac{\frac{1}{1+\left| \frac{a_1+2b_1+2c_1+d_1}{6} - \frac{a_1+2b_1+2c_1+d_1}{6} \right|} \cdot }{\frac{2w_j(\eta_A^1(x_j).\eta_A^1(x_j)+\eta_A^2(x_j).\eta_A^2(x_j)+....+\eta_A^p(x_j)\eta_A^p(x_j))}{(\eta_A^1)^2(x_j)+(\eta_A^1)^2(x_j)+(\eta_A^2)^2(x_j)+(\eta_A^2)^2(x_j)+...+(\eta_A^P)^2(x_j)+(\eta_A^P)^2(x_j)}} \right)$$

$$= \sum_{j=1}^{n} \left(\frac{1}{1+0} \cdot \frac{2w_j\left((\eta_A^1)^2(x_j)+(\eta_A^2)^2(x_j)+....+(\eta_A^p)^2(x_j)\right)}{2\left((\eta_A^1)^2(x_j)+(\eta_A^2)^2(x_j)+....+(\eta_A^p)^2(x_j)\right)} \right)$$

$$= 1.$$

4. ITFM-number Multi-criteria Decision Making Method

In this section, we define ITFMN-multi-criteria decision making method based on Dice vector similarity measure for intuitionistic trapezoidal fuzzy multi-numbers.

Definition 4.1 Let $U = (u_1, u_2, ..., u_m)$ be a set of alternatives, $A = (a_1, a_2, ..., a_n)$ be the set of criteria, $w = (w_1, w_2, ..., w_n)^T$ be the weight vector of the $a_j (j = 1,2 ..., n)$ such that

$w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$ and $\left[b_{ij} \right]_{mxn} = \left\langle \left[a_{ij}, b_{ij}, c_{ij}, d_{ij} \right]; \left(\eta_{ij}^1, \eta_{ij}^2, ..., \eta_{ij}^p \right), \left(v_{ij}^1, v_{ij}^2, ..., v_{ij}^p \right) \right\rangle$ be

the decision matrix in which the rating values of the alternatives. Then

$$
\left[b_{ij} \right]_{m \times n} =
\begin{matrix}
 & \begin{matrix} a_1 & a_2 & \cdots & a_n \end{matrix} \\
\begin{matrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{matrix} &
\begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1n} \\
b_{21} & b_{22} & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
b_{m1} & b_{m2} & \cdots & b_{mn}
\end{pmatrix}
\end{matrix}
$$

is called a ITFM-number multi-criteria decision making matrix of the decision maker.

Also; r^+ is positive ideal ITFM-numbers solution of decision matrix $\left[b_{ij} \right]_{mxn}$ as form:

$$
r^+ = \left\langle [1,1,1,1], (1,1,...,1), (0,0,...,0) \right\rangle
$$

and r^- is negative ideal ITFM-numbers solution of decision matrix $\left[b_{ij} \right]_{mxn}$ as form:

$$
r^- = \left\langle [1,1,1,1], (0,0,...,0), (1,1,...,1) \right\rangle
$$

Algorithm:

Step1. Give the decision-making matrix $\left[b_{ij} \right]_{mxn}$ for decision;

Step2. Calculate the weighted Dice vector similarity S_i between positive ideal (or negative ideal) ITFMN solution r^+ and $u_i = \left\langle \left[a_{ij}, b_{ij}, c_{ij}, d_{ij} \right]; \left(\eta_{ij}^1, \eta_{ij}^2, ..., \eta_{ij}^p \right), \left(v_{ij}^1, v_{ij}^2, ..., v_{ij}^p \right) \right\rangle$ and $(i = 1, 2, ..., m)$ as;

$$
D_w(\overline{A}, \overline{B}) = \frac{1}{n} \sum_{j=1}^{n} \left(\frac{\frac{1}{1 + d(\overline{A}, \overline{B})} \cdot}{\frac{2.w_j.(\eta_A^1(x_j).\eta_B^1(x_j) + ... + \eta_A^p(x_j).\eta_B^p(x_j)).(v_A^1(x_j).v_B^1(x_j) + ... + v_A^p(x_j).v_B^p(x_j))}{\left((\eta_A^1)^2(x_j) + ... + (\eta_A^p)^2(x_j) + (\eta_B^p)^2(x_j) \right).\left((v_A^1)^2(x_j) + ... + (v_A^p)^2(x_j) + (v_B^p)^2(x_j) \right)}} \right)
$$

Step3. Determine the non-increasing order of $S_i = D_{w_i}(u_i, r^+)$ $(i = 1, 2, ..., m)$, $(j = 1, 2, ..., n)$

Step4. Select the best alternative.

Now, we give a numerical example as follows;

Example 4.2 Architecture means the design of structures. It means designing and shaping structures in a way. It requires great imagination. Then it should be transferred to paper. At this stage, there may be some difficulties, and in terms of time and design, it will be difficult to put the design literally on paper. So it would be best to use computer-aided programs. Let's consider the entrance door of Gaziantep Zoo (Figure-1) drawn by Dr. Derya BAKBAK [39].
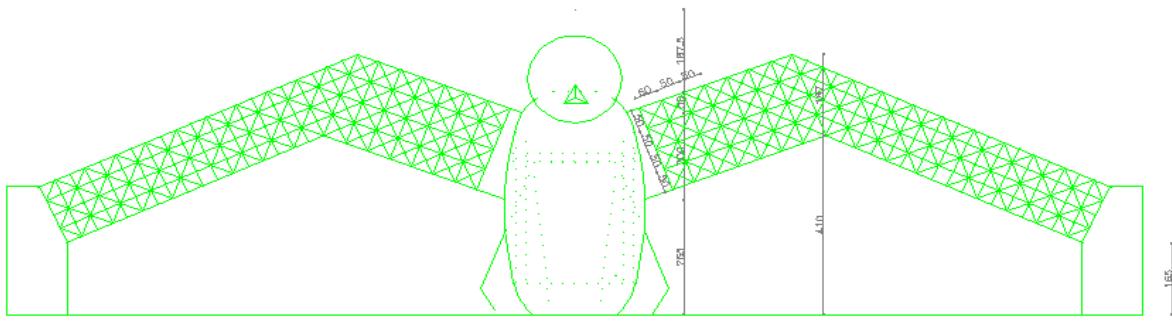
Figure-1: Entrance Door of Gaziantep Zoo

Ezgi Architecture Company wants to choose computer-aided programs that will draw similar shapes to the entrance gate of Gaziantep zoo. Therefore, Ezgi Architecture Company wants to buy the best of four computer-aided programs. Four types of programs (alternatives) $u_i(i = 1,2,3,4)$ are available. The Ezgi architecture company takes into account two attributes to evaluate the alternatives; c_1 =2D; c_2 =3D use the ITFMN values to evaluate the four possible alternatives $u_i(i = 1, 2, 3, 4)$ under the above two attributes. Also, the weight vector of the attributes $c_j(j = 1,2)$ is $\omega = (0.2, 0.5, 0.1, 0.2)^T$. Then,

Algorithm

Step1. Constructed the decision matrix provided by the Ezgi Architecture Company as;

Table 2: Decision matrix given by Ezgi Architecture Company

	c_1	c_2
u_1	$\langle[0.3,0.5,0.7,0.9];(0.4,0.5,0.6),(0.2,0.5,0.8)\rangle$	$\langle[0.6,0.7,0.8,0.9];(0.1,0.4,0.7),(0.3,0.4,0.5)\rangle$
u_2	$\langle[0.2,0.3,0.5,0.6];(0.2,0.3,0.8),(0.1,0.4,0.6)\rangle$	$\langle[0.1,0.3,0.4,0.5];(0.2,0.4,0.7),(0.2,0.5,0.8)\rangle$
u_3	$\langle[0.1,0.4,0.5,0.6];(0.3,0.4,0.5),(0.2,0.3,0.7)\rangle$	$\langle[0.3,0.4,0.5,0.6];(0.1,0.4,0.7),(0.2,0.5,0.8)\rangle$
u_4	$\langle[0.2,0.5,0.6,0.7];(0.3,0.4,0.6),(0.2,0.1,0.3)\rangle$	$\langle[0.2,0.3,0.5,0.6];(0.1,0.2,0.5),(0.2,0.3,0.4)\rangle$

Step2. Computed the positive ideal ITFM-numbers solution as;

$$r^+ = \left\langle [1,1,1,1], (1,1,...,1), (0,0,...,0) \right\rangle$$

Step3. Calculated the weighted Dice vector similarity measures, $S_i = D_{w_i}(\mathbf{u}_i, r^+)$ as;

The Proposed method	Measure value	Ranking order
$S_i = D_{w_i}(\mathbf{u}_i, r^+)$	$D_{w_i}(\mathbf{u}_1, r^+) = 0,1302$	
	$D_{w_i}(\mathbf{u}_2, r^+) = 0,3341$	
	$D_{w_i}(\mathbf{u}_3, r^+) = 0,0685$	$S_2 > S_1 > S_4 > S_3$
	$D_{w_i}(\mathbf{u}_4, r^+) = 0,1197$	

Step4. So the Ezgi architecture company will select the computer-aided program $\mathbf{u}_2$. In any case if they do not want to choose $\mathbf{u}_2$ due to some reasons they second choice will be $\mathbf{u}_1$.

5. Conclusions

In this paper, we developed a multi-criteria decision making for intuitionistic trapezoidal fuzzy multi-number based on weighted Dice vector similarity measures and applied to a numerical example in order to confirm the practicality and accuracy of the proposed method. In the future, the method can be extend with different similarity and distance measures in neutrosophic set.

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Chapter Twelve

Improved Hybrid Vector Similarity Measures And Their Applications on Trapezoidal Fuzzy Multi Numbers

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Abstract

In this chapter, we put forward some similarity measures for Trapezoidal Fuzzy Multi Numbers (TFMN) such as; Jaccard similarity measure, weighted Jacard similarity measure, Cosine similarity measure, weighted cosine similarity measure, Hybrid vector similarity measure and weighted Hybrid vector similarity measure. Also we investigate the propositions of the similarity measures. Moreover, a multi-criteria decision-making method for TFMN is improved based on these given similarity measures. Then, a practical example is shown to approve the feasibility of the new method. As a result, we compare the proposed method with the existing methods in order to show the effectiveness and efficiency of the developed method in this study.

Keywords: Trapezoidal Fuzzy Multi Numbers, Jacard similarity measure, Cosine similarity measure, Hybrid vector similarity measure, Decision making.

1.Introduction

Multi attribute decision making has got much interest to the investigators because it has obtained excellent admission in the fields of operations research, engineering, and management, signal processing etc. We see multi attribute decision making problems under a lot of conditions, in which the number of possible options and actions need to be selected based on a set of predefined attributes. Many of research works have been done on multi attribute decision making problems, in which the ratings of alternatives and/or attribute values are explained in terms of crisp numbers such as interval numbers, fuzzy numbers, interval-valued fuzzy numbers, intuitionistic fuzzy numbers, interval-valued intuitionistic fuzzy numbers, etc. But, in realistic conditions, because of time pressure, complexity of the problem, lack of information processing capabilities, poor knowledge of the public domain and information, decision makers cannot provide exact evaluation of decision-parameters involved in multi attribute decision making problems. In such situation, preference information of alternatives with respect to the attributes provided by the decision makers may be imprecise or incomplete in nature. In the study, we suggest Jaccard vector similarity measures for trapezoidal fuzzy multi numbers and cosine vector similarity measure for trapezoidal fuzzy multi numbers by extending the concept of studied in [10] and [11] to trapezoidal fuzzy multi numbers and establish some of their basic properties. Notions of similarity, decision making, measure and algebraic etc. of neutrosophic sets have been introduces and their applications in several areas [12-33]. And we also proposed Hybrid vector similarity measure for trapezoidal fuzzy multi numbers and establish some of their basic properties. Additionally we also show the application of

these suggested similarity measures. In order to do so, the rest of the chapter is organized as follows: Section 2 presents the preliminaries of fuzzy set, fuzzy number, multi-fuzzy set and similarity measures including Jaccard, Cosine and Hybrid. Section 3 represents some similarity measures for TFMNs including Jaccard similarity measure and Cosine similarity measure. Section 4 is devoted to develop the hybrid vector similarity measures for TFMNs. Medical diagnosis using the Jaccard, Cosine and Hybrid similarity measures is described in Section 5 and compared the results with other existing methods to demonstrate the effectiveness of the proposed similarity measures. Finally in Section 6, we proposed conclusions for the effectiveness and efficiency with similarity.

2.PRELIMINARY

In this section, we proposed some basic concepts related to fuzzy set, fuzzy number, multi-fuzzy set and similarity measures for TFMN's including Jaccard similarity measure, Cosine similarity measure which will be used in the next sections.

Definition 2.1[1]Let X be a non-empty set. A fuzzy set F on X is defined as:

$$F = \{\langle x, \mu_F(x) \rangle : x \in X\} \text{ where } \mu_F : X \to [0,1] \text{ for } x \in X .$$

Definition 2.2[2] t-norms are associative, monotonic and commutative two valued functions t that map from $[0,1] \times [0,1]$ into $[0,1]$. These properties are formulated with the following conditions:

1. $t(0,0) = 0$ and $t(\mu_{x_1}(x),1) = t(1, \mu_{x_1}(x)) = \mu_{x_1}(x)$

2. If $\mu_{x_1}(x) \leq \mu_{x_3}(x)$ and $\mu_{x_2}(x) \leq \mu_{x_4}(x)$ then $t(\mu_{x_1}(x), \mu_{x_2}(x)) \leq t(\mu_{x_3}(x), \mu_{x_4}(x))$,

3. $t(\mu_{x_1}(x), \mu_{x_2}(x)) = t(\mu_{x_2}(x), \mu_{x_1}(x))$,

4. $t(\mu_{x_1}(x), t(\mu_{x_2}(x), \mu_{x_3}(x))) = t(t(\mu_{x_1}(x), \mu_{x_2}(x)), \mu_{x_3}(x))$

Definition 2.3[2] s-norms are associative, monotonic and commutative two placed functions s which map from $[0,1] \times [0,1]$ into $[0,1]$. These properties are formulated with the following conditions:

1. $s(1,1) = 1$ and $s(\mu_{x_1}(x),0) = s\left(0, \mu_{x_1}(x)\right) = \mu_{x_1}(x)$,

2. If $\mu_{x_1}(x) \leq \mu_{x_3}(x)$ and $\mu_{x_2}(x) \leq \mu_{x_4}(x)$, then $s(\mu_{x_1}(x), \mu_{x_2}(x)) \leq s(\mu_{x_3}(x), \mu_{x_4}(x))$,

3. $s(\mu_{x_1}(x), \mu_{x_2}(x)) = s(\mu_{x_2}(x), \mu_{x_1}(x))$,

4. $s(\mu_{x_1}(x), s(\mu_{x_2}(x), \mu_{x_3}(x))) = s(s(\mu_{x_1}(x), \mu_{x_2}(x)), \mu_{x_3}(x))$.

t-norm and t-conorm is related in a sense of logical duality. Typical dual pairs of non-parametrized t-norm and t-conorm are compiled below:

1. Drastic product: $t_w(\mu_{x_1}(x), \mu_{x_2}(x)) = \begin{cases} \min\{\mu_{x_1}(x), \mu_{x_2}(x)\}, & \max\{\mu_{x_1}(x), \mu_{x_2}(x)\} = 1 \\ 0 & otherwise \end{cases}$

2. Drastic sum: $s_w(\mu_{x_1}(x), \mu_{x_2}(x)) = \begin{cases} \max\{\mu_{x_1}(x), \mu_{x_2}(x)\}, & \min\{\mu_{x_1}(x), \mu_{x_2}(x)\} = 0 \\ 1 & otherwise \end{cases}$

3. Bounded product:

$t_1(\mu_{x_1}(x), \mu_{x_2}(x)) = \max\{0, \mu_{x_1}(x) + \mu_{x_2}(x) - 1\}$

4. Bounded sum:

$s_1(\mu_{x_1}(x), \mu_{x_2}(x)) = \min\{1, \mu_{x_1}(x) + \mu_{x_2}(x)\}$

5. Einstein product:

$t_{1.5}(\mu_{x_1}(x), \mu_{x_2}(x)) = \dfrac{\mu_{x_1}(x).\mu_{x_2}(x)}{2 - [\mu_{x_1}(x) + \mu_{x_2}(x) - \mu_{x_1}(x).\mu_{x_2}(x)]}$

6. Einstein sum:

$s_{1.5}(\mu_{x_1}(x), \mu_{x_2}(x)) = \dfrac{\mu_{x_1}(x) + \mu_{x_2}(x)}{1 + \mu_{x_1}(x).\mu_{x_2}(x)}$

7. Algebraic product:

$t_2(\mu_{x_1}(x), \mu_{x_2}(x)) = \mu_{x_1}(x).\mu_{x_2}(x)$

8. Algebraic sum:

$s_2(\mu_{x_1}(x), \mu_{x_2}(x)) = \mu_{x_1}(x) + \mu_{x_2}(x) - \mu_{x_1}(x).\mu_{x_2}(x)$

9. Hamacher product:

$$t_{2.5}(\mu_{x_1}(x), \mu_{x_2}(x)) = \frac{\mu_{x_1}(x).\mu_{x_2}(x)}{\mu_{x_1}(x) + \mu_{x_2}(x) - \mu_{x_1}(x).\mu_{x_2}(x)}$$

10. Hamacher Sum:

$$s_{2.5}(\mu_{x_1}(x), \mu_{x_2}(x)) = \frac{\mu_{x_1}(x) + \mu_{x_2}(x) - 2.\mu_{x_1}(x).\mu_{x_2}(x)}{1 - \mu_{x_1}(x).\mu_{x_2}(x)}$$

11. Minimum:

$$t_3(\mu_{x_1}(x), \mu_{x_2}(x)) = \min\{\mu_{x_1}(x), \mu_{x_2}(x)\}$$

12. Maximum:

$$s_3(\mu_{x_1}(x), \mu_{x_2}(x)) = \max\{\mu_{x_1}(x), \mu_{x_2}(x)\}$$

Definition 2.4[3] Let X be a non-empty set. A multi-fuzzy set G on X is defined as $G = \{\langle x, \mu_G^1(x), \mu_G^2(x), ..., \mu_G^i(x), ... \rangle : x \in X\}$ where $\mu_G^i : X \to [0,1]$ for all i $\in \{1, 2, ..., p\}$ and $x \in X$

Definition 2.5[9] Let $\eta_A^i \in [0,1]$ (i $\in \{1, 2, ..., p\}$) and $a, b, c, d \in \mathbb{R}$ such that $a \le b \le c \le d$. Then, a trapezoidal fuzzy multi-number (TFM number) $\tilde{a} = \langle (a, b, c, d); \eta_A^1, \eta_A^2, ..., \eta_A^p \rangle$ is a special fuzzy multi-set on the real number set $\mathbb{R}$, whose membership functions are defined as

$$\mu_A^i(x) = \begin{cases} (x - a_1)\eta_A^i / (b_1 - a_1) & a_1 \le x \le b_1 \\ \eta_A^i & b_1 \le x \le c_1 \\ (d_1 - x)\eta_A^i / (d_1 - c_1) & c_1 \le x \le d_1 \\ 0 & otherwise \end{cases}$$

Note that the set of all TFM-number on $\mathbb{R}$ will be denoted by Λ.

Definition 2.6[9] Let $A = \langle (a_1, b_1, c_1, d_1); \eta_A^1, \eta_A^2, ..., \eta_A^p \rangle$, $B = \langle (a_2, b_2, c_2, d_2); \eta_B^1, \eta_B^2, ..., \eta_B^p \rangle$ $\in \Lambda$ and $\gamma \ne 0$ be any real number. Then,

1. $A + B = \langle (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2); s(\eta_A^1, \eta_B^1), s(\eta_A^2, \eta_B^2), ..., s(\eta_A^P, \eta_B^P) \rangle$

2. $A - B = \left\langle \left(a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2\right); s(\eta_A^1, \eta_B^1), s(\eta_A^2, \eta_B^2), ..., s(\eta_A^P, \eta_B^P) \right\rangle$

3. $A.B = \begin{cases} \left\langle \begin{array}{l} \left(a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2\right); \\ t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \end{array} \right\rangle & (d_1 > 0, d_2 > 0) \\ \left\langle \begin{array}{l} \left(a_1 d_2, b_1 c_2, c_1 b_2, d_1 a_2\right); \\ t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \end{array} \right\rangle & (d_1 < 0, d_2 > 0) \\ \left\langle \begin{array}{l} \left(d_1 d_2, c_1 c_2, b_1 b_2, a_1 a_2\right); \\ t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \end{array} \right\rangle & (d_1 < 0, d_2 < 0) \end{cases}$

4. $A / B = \begin{cases} \left\langle \begin{array}{l} \left(a_1 / d_2, b_1 / c_2, c_1 / b_2, d_1 / a_2\right); \\ t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \end{array} \right\rangle & (d_1 > 0, d_2 > 0) \\ \left\langle \begin{array}{l} \left(d_1 / d_2, c_1 / c_2, b_1 / b_2, a_1 / a_2\right); \\ t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \end{array} \right\rangle & (d_1 < 0, d_2 > 0) \\ \left\langle \begin{array}{l} \left(d_1 / a_2, c_1 / b_2, b_1 / c_2, a_1 / d_2\right); \\ t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \end{array} \right\rangle & (d_1 < 0, d_2 < 0) \end{cases}$

5. $\gamma A = \left\langle \left(\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1\right); 1 - (1 - \eta_A^1)^\gamma, 1 - (1 - \eta_A^2)^\gamma, ..., 1 - (1 - \eta_A^p)^\gamma \right\rangle (\gamma \geq 0)$

6. $A^\gamma = \left\langle \left(a_1^\gamma, b_1^\gamma, c_1^\gamma, d_1^\gamma\right); (\eta_A^1)^\gamma, (\eta_A^2)^\gamma, ..., (\eta_A^P)^\gamma \right\rangle (\gamma \geq 0)$

Definition 2.7[9] Let $A = \left\langle \left(a_1, b_1, c_1, d_1\right) : \eta_A^1, \eta_A^2, ..., \eta_A^p \right\rangle \in \Lambda$, Then, normalized TFM-number of A is given by

$$\bar{A} = \left\langle \left(\frac{a_1}{a_1 + b_1 + c_1 + d_1}, \frac{b_1}{a_1 + b_1 + c_1 + d_1}, \frac{c_1}{a_1 + b_1 + c_1 + d_1}, \frac{d_1}{a_1 + b_1 + c_1 + d_1} \right); \eta_A^1, \eta_A^2, ..., \eta_A^p \right\rangle$$

Definition 2.8: Let $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ be the two vectors of length n where all the coordinates are positive. The Jaccard index of these two vectors (measuring the "similarity" of these vectors) (Jaccard 1901) is defined as

$$J = \frac{X.Y}{[\![X]\!]_2^2 + [\![Y]\!]_2^2 - X.Y} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{n} x_i y_i} \qquad [11]$$

where $X.Y = \sum_{I=1}^{n} x_i y_i$ is the inner product of the vectors X and Y and where

$[\![X]\!]_2 = \sqrt{\sum_{I=1}^{n} x^2}$ and $[\![Y]\!]_2 = \sqrt{\sum_{I=1}^{n} y^2}$ are the Euclidean norms of X and Y (also called the L2

norms).

A cosine formula (Salton and McGill 1987) is then defined as the inner product of these two vectors divided by the product of their lengths. This is nothing but the cosine of the angle between the vectors. The cosine measure is defined as

$$\text{Cos} = \frac{X.Y}{[\![X]\!]_2 . [\![Y]\!]_2} = \frac{\sum_{I=1}^{n} x_i y_i}{\sqrt{\sum_{i=1}^{n} x_i^2} + \sqrt{\sum_{i=1}^{n} y_i^2}} \qquad [10]$$

These two formulas are similar in the sense that they take values in the interval [0,l]. Jaccard formula are undefined if $x_i = y_i = 0$ holds for all the $i, (i = 1,2,...,n)$, and then we let the this measure value be zero when $x_i = y_i = 0$ holds for all the $i, (i = 1,2,...,n)$. However, the cosine formula is undefined if $x_i = 0$ and/or $y_i = 0$ holds for all the $i, (i = 1,2,...,n)$, and then we let the cosine measure value be zero when $x_i = 0$ and/or $y_i = 0$ holds for all the $i, (i = 1,2,...,n)$.

3. JACCARD SIMILARITY MEASURE AND COSINE SIMILARITY MEASURE FOR TRAPEZOIDAL FUZZY MULTI NUMBERS

In this section, we introduced some similarity measures for TFMNs including Jaccard similarity measure and Cosine similarity measure.

Definition 3.1: Let $a_1 < a_2 < a_3 < a_4$, $a_1, a_2, a_3, a_4 \in R$, $A = \left\langle (a_1, a_2, a_3, a_4); \eta_A^1, \eta_A^2, ..., \eta_A^p \right\rangle$, $B = \left\langle (b_1, b_2, b_3, b_4); \eta_B^1, \eta_B^2, ..., \eta_B^p \right\rangle$ be two TFMNs in the set of real numbers $\mathbb{R}$. Then; Jaccard similarity measure between TFMN A and B denoted $J(A, B)$ is defined as;

$$J(\text{A}, \text{B}) = \frac{1}{p} \sum_{k=1}^{p} \left(\left(1 - \frac{\sum_{j=1}^{4} |a_j^k - b_j^k|}{4} \right) \cdot \frac{\sum_{i=1}^{p} (\eta_A^i)_k . (\eta_B^i)_k}{\sum_{i=1}^{p} (\eta_A^i)_k^2 + \sum_{i=1}^{p} (\eta_B^i)_k^2 - \sum_{i=1}^{p} (\eta_A^i)_k . (\eta_B^i)_k} \right)$$

Note: Let $A = \left\langle (a_1, a_2, a_3, a_4); \eta_A^1, \eta_A^2, ..., \eta_A^p \right\rangle$, be a trapezoidal fuzzy multi number. $a_1 < a_2 < a_3 < a_4$ and $a_1, a_2, a_3, a_4 \in \mathbb{R}$ if $a_2 = a_3$ then this trapezoidal fuzzy multi number turns to triangular fuzzy multiple number.

Proposition 3.2 Let $J(A,B)$ be a Jaccard similarity measure between TFMN's A and B. Then we have,

i. $0 \leq J(A,B) \leq 1$

ii. $J(A,B) = J(B,A)$

iii. $J(A,B) = 1$ *for* $A = B$, i.e. $\eta_A^i = \eta_B^i$, (i=1,2...,p)

Proof 3.3

i. it is clear from Definition 3.1

ii.
$$J(A,B) = \frac{1}{p}\sum_{k=1}^{p}\left(\left(1 - \frac{\sum_{j=1}^{4}\left|a_j^k - b_j^k\right|}{4}\right) \cdot \frac{\sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}{\sum_{i=1}^{p}(\eta_A^i)_k^{2} + \sum_{i=1}^{p}(\eta_B^i)_k^{2} - \sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}\right)$$

$$= \frac{1}{p}\sum_{k=1}^{p}\left(\left(1 - \frac{\sum_{j=1}^{4}\left|a_j^k - b_j^k\right|}{4}\right) \cdot \frac{\sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}{\sum_{i=1}^{p}(\eta_A^i)_k^{2} + \sum_{i=1}^{p}(\eta_A^i)_k^{2} - \sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}\right)$$

$$= \frac{1}{p}\sum_{k=1}^{p}\left(\left(1 - \frac{\sum_{j=1}^{4}\left|b_j^k - a_j^k\right|}{4}\right) \cdot \frac{\sum_{i=1}^{p}(\eta_B^i)_k.(\eta_A^i)_k}{\sum_{i=1}^{p}(\eta_B^i)_k^{2} + \sum_{i=1}^{p}(\eta_A^i)_k^{2} - \sum_{i=1}^{p}(\eta_B^i)_k.(\eta_A^i)_k}\right)$$

$$= J(B,A).$$

iii.
$$J(A,B) = \frac{1}{p}\sum_{k=1}^{p}\left(\left(1 - \frac{\sum_{j=1}^{4}\left|a_j^k - b_j^k\right|}{4}\right) \cdot \frac{\sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}{\sum_{i=1}^{p}(\eta_A^i)_k^{2} + \sum_{i=1}^{p}(\eta_B^i)_k^{2} - \sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}\right)$$

$$= \frac{1}{p}\sum_{k=1}^{p}\left(\left(1 - \frac{\left|a_1^k - b_1^k\right| + \left|a_2^k - b_2^k\right| + \left|a_3^k - b_3^k\right| + \left|a_4^k - b_4^k\right|}{4} \cdot \frac{(\eta_A^1)_k.(\eta_B^1)_k + (\eta_A^2)_k.(\eta_B^2)_k + ... + (\eta_A^p)_k.(\eta_B^p)_k}{\sum_{i=1}^{p}(\eta_A^i)_k^{2} + \sum_{i=1}^{p}(\eta_B^i)_k^{2} - \sum_{i=1}^{p}(\eta_A^i)_k^{2}.(\eta_B^i)_k^{2}}\right)\right)$$

$$= \frac{1}{p}\sum_{k=1}^{p}\left(\left(1 - \frac{\left|a_1^k - a_1^k\right| + \left|a_2^k - a_2^k\right| + \left|a_3^k - a_3^k\right| + \left|a_4^k - a_4^k\right|}{4} \cdot \frac{(\eta_A^1)_k.(\eta_A^1)_k + (\eta_A^2)_k.(\eta_A^2)_k + ... + (\eta_A^p)_k.(\eta_A^p)_k}{(\eta_A^1)_k^2 + ... + (\eta_A^p)_k^2 + (\eta_A^1)_k^2 + ... + (\eta_A^p)_k^2 - ((\eta_A^1)_k^2 + ...(\eta_A^p)_k^2)}\right)\right)$$

$$= \frac{1}{p}\sum_{k=1}^{p}\left(\left(1 - 0. \frac{(\eta_A^1)_k.(\eta_A^1)_k + (\eta_A^2)_k.(\eta_A^2)_k + ... + (\eta_A^p)_k.(\eta_A^p)_k}{(\eta_A^1)_k^2 + ... + (\eta_A^p)_k^2 + (\eta_A^1)_k^2 + ... + (\eta_A^p)_k^2 - ((\eta_A^1)_k^2 + ...(\eta_A^p)_k^2)}\right)\right)$$

$$= \frac{1}{p}\sum_{k=1}^{p}\left(\frac{(\eta_A^1)_k^2 + (\eta_A^2)_k^2 + ... + (\eta_A^p)_k^2}{(\eta_A^1)_k^2 + (\eta_A^2)_k^2 + ... + (\eta_A^p)_k^2}\right)$$

$$=1.$$

Example 3.4 Let $J(A,B)$ be a Jaccard similarity measure between TFMN's $A = \langle(1,2,3,5);0.2,0.4,0.5,0.7\rangle$ and $B = \langle(2,3,4,5);0.3,0.1,0.6,0.4\rangle$.

$$J(A,B) = \frac{1}{p}\sum_{k=1}^{p}\left(\left(1 - \frac{\sum_{j=1}^{4}\left|a_j^k - b_j^k\right|}{4}\right). \frac{\sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}{\sum_{i=1}^{p}(\eta_A^i)_k^2 + \sum_{i=1}^{p}(\eta_B^i)_k^2 - \sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}\right)$$

$$= \left(1 - \frac{|1-2| + |2-3| + |3-4| + |5-5|}{4}. \frac{(0,2.0,3 + 0,4.0,1 + 0,5.0,6 + 0,7.0,4)}{(0,2^2 + 0,4^2 + 0,5^2 + 0,7^2) + (0,3^2 + 0,1^2 + 0,6^2 + 0,4^2) - (0,2.0,3 + 0,4.0,1 + 0,5.0,6 + 0,7.0,4)}\right)$$

$$= \left(1 - \frac{3}{4}. \frac{(0,06 + 0,04 + 0,30 + 0,28)}{(0,04 + 0,16 + 0,25 + 0,49) + (0,09 + 0,01 + 0,36 + 0,16) - (0,06 + 0,04 + 0,30 + 0,28)}\right)$$

$$= \left(\frac{1}{4}. \frac{0.68}{0.94 + 0,62 - 0,68}\right) = \frac{0,68}{3,52} \cong 0,193$$

Definition 3.5 Let $A = \langle(a_1,a_2,a_3,a_4);\eta_A^1,\eta_A^2,...,\eta_A^p\rangle$, $B = \langle(b_1,b_2,b_3,b_4);\eta_B^1,\eta_B^2,...,\eta_B^p\rangle$ be

two TFMNs in the set of real numbers $\mathbb{R}$ and $w_r \in [0,1]$ be the weight of each element for

$r = (1, 2, ..., n)$ such that $\sum_{r=1}^{n} w_r = 1$. Then; Jaccard similarity measure between TFMN A and

B denoted $J_w(A, B)$ is defined as;

$$J_w(A, B) = \frac{1}{p} \sum_{r=1}^{n} \sum_{k=1}^{p} \left(\left(1 - \frac{\sum_{j=1}^{4} |a_j^k - b_j^k|}{4} \right) \cdot \frac{w_r \cdot \sum_{i=1}^{p} (\eta_A^i)_k \cdot (\eta_B^i)_k}{\sum_{i=1}^{p} (\eta_A^i)_k^2 + \sum_{i=1}^{p} (\eta_B^i)_k^2 - \sum_{i=1}^{p} (\eta_A^i)_k \cdot (\eta_B^i)_k} \right)$$

Proposition 3.6 Let $J_w(A, B)$ be a weighted Jaccard similarity measure between TFMN's A and B, $w_r \in [0,1]$ be the weight of each element for $r = (1, 2, ..., n)$ such that $\sum_{r=1}^{n} w_r = 1$ Then weighted Jaccard vector similarity measure between TFMN's A and B;

i. $0 \le J_w(A, B) \le 1$

ii. $J_w(A, B) = J_w(B, A)$

iii. $J_w(A, B) = 1$ *for* $A = B$ i.e. ($\eta_A^1 = \eta_B^1, \eta_A^2 = \eta_B^2, ..., \eta_A^p = \eta_B^p$)

Proof 3.7

i. it is clear from Definition 3.5

ii. $J_w(A, B) = \frac{1}{p} \sum_{r=1}^{n} \sum_{k=1}^{p} \left(\left(1 - \frac{\sum_{j=1}^{4} |a_j^k - b_j^k|}{4} \right) \cdot \frac{w_r \cdot \sum_{i=1}^{p} (\eta_A^i)_k \cdot (\eta_B^i)_k}{\sum_{i=1}^{p} (\eta_A^i)_k^2 + \sum_{i=1}^{p} (\eta_B^i)_k^2 - \sum_{i=1}^{p} (\eta_A^i)_k \cdot (\eta_B^i)_k} \right)$

$$= \frac{1}{p} \sum_{r=1}^{n} \sum_{k=1}^{p} \left(\left(1 - \frac{\sum_{j=1}^{4} |a_j^k - b_j^k|}{4} \right) \cdot \frac{w_r \cdot \sum_{i=1}^{p} (\eta_A^i)_k \cdot (\eta_B^i)_k}{\sum_{i=1}^{p} (\eta_A^i)_k^2 + \sum_{i=1}^{p} (\eta_B^i)_k^2 - \sum_{i=1}^{p} (\eta_A^i)_k \cdot (\eta_B^i)_k} \right)$$

$$= \frac{1}{p} \sum_{r=1}^{n} \sum_{k=1}^{p} \left(\left(1 - \frac{\sum_{j=1}^{4} |b_j^k - a_j^k|}{4} \right) \cdot \frac{w_r \cdot \sum_{i=1}^{p} (\eta_B^i)_k \cdot (\eta_A^i)_k}{\sum_{i=1}^{p} (\eta_B^i)_k^2 + \sum_{i=1}^{p} (\eta_A^i)_k^2 - \sum_{i=1}^{p} (\eta_B^i)_k \cdot (\eta_A^i)_k} \right)$$

$$= J_w(B, A).$$

iii. $J_w(A,B) = \dfrac{1}{p}\sum\limits_{r=1}^{n}\sum\limits_{k=1}^{p}\left(\left(1 - \dfrac{\sum\limits_{j=1}^{4}\left|a_j^k - b_j^k\right|}{4}\right)\cdot\dfrac{w_r\cdot\sum\limits_{i=1}^{p}(\eta_A^i)_k\cdot(\eta_B^i)_k}{\sum\limits_{i=1}^{p}(\eta_A^i)_k^2 + \sum\limits_{i=1}^{p}(\eta_B^i)_k^2 - \sum\limits_{i=1}^{p}(\eta_A^i)_k\cdot(\eta_B^i)_k}\right).$

$= \dfrac{1}{p}\sum\limits_{r=1}^{n}\sum\limits_{k=1}^{p}\left(\left(1 - \dfrac{\left|a_1^k - b_1^k\right| + \left|a_2^k - b_2^k\right| + \left|a_3^k - b_3^k\right| + \left|a_4^k - b_4^k\right|}{4}\right)\cdot\dfrac{w_r\cdot[(\eta_A^1)_k\cdot(\eta_B^1)_k + (\eta_A^2)_k\cdot(\eta_B^2)_k + \dots + (\eta_A^p)_k\cdot(\eta_B^p)_k]}{(\eta_A^1)_k^2 + \dots + (\eta_A^p)_k^2 + (\eta_B^1)_k^2 + \dots + (\eta_B^p)_k^2 - ((\eta_A^1)_k\cdot(\eta_B^1)_k + \dots(\eta_A^p)_k\cdot(\eta_B^p)_k)}\right)$

$= \dfrac{1}{p}\sum\limits_{r=1}^{n}\sum\limits_{k=1}^{p}\left(\left(1 - \dfrac{\left|a_1^k - a_1^k\right| + \left|a_2^k - a_2^k\right| + \left|a_3^k - a_3^k\right| + \left|a_4^k - a_4^k\right|}{4}\right)\cdot\dfrac{w_r\cdot[(\eta_A^1)_k\cdot(\eta_A^1)_k + (\eta_A^2)_k\cdot(\eta_A^2)_k + \dots + (\eta_A^p)_k\cdot(\eta_A^p)_k]}{(\eta_A^1)_k^2 + \dots + (\eta_A^p)_k^2 + (\eta_A^1)_k^2 + \dots + (\eta_A^p)_k^2 - ((\eta_A^1)_k^2 + \dots(\eta_A^p)_k^2)}\right)$

$= \dfrac{1}{p}\sum\limits_{r=1}^{n}\sum\limits_{k=1}^{p}\left((1-0)\cdot\dfrac{w_r\cdot[(\eta_A^1)_k^2 + \dots + (\eta_A^p)_k^2]}{(\eta_A^1)_k^2 + \dots + (\eta_A^p)_k^2 + (\eta_A^1)_k^2 + \dots + (\eta_A^p)_k^2 - ((\eta_A^1)_k^2 + \dots(\eta_A^p)_k^2)}\right)$

$= \dfrac{1}{p}\sum\limits_{r=1}^{n}\sum\limits_{k=1}^{p}\left(\dfrac{w_r\cdot[(\eta_A^1)_k^2 + \dots + (\eta_A^p)_k^2]}{(\eta_A^1)_k^2 + \dots + (\eta_A^p)_k^2}\right)$

$= \sum\limits_{r=1}^{n}w_r$

$=1.$

Example 3.8 Let $A = \langle(1.2.3.4);0.1,0.3,0.4,0.5\rangle$, $B = \langle(1,2,3,5);0.2,0.3,0.5,0.6\rangle$ be two

TFMNs in the set of real numbers $\mathbb{R}$ and W_i be the weight of each element for $r = (1,2)$

$w_1 = 0.3, w_2 = 0.7$ such that $\sum\limits_{r=1}^{n}w_r = 1$. Then; Jaccard similarity measure between TFMN A

and B is;

$$J_w(A,B) = \left(\begin{array}{c} 1 - \dfrac{|1-1|+|2-2|+|3-3|+|4-5|}{4} \cdot \\[3mm] \dfrac{0,3.[(0,1.0,2+0,3.0,3+0,4.0,5+0,5.0,6]}{(0,1^2+0,3^2+0,4^2+0,5^2)+(0,2^2+0,3^2+0,5^2+0,6^2)-(0,1.0,2+0,3.0,3+0,4.0,5+0,5.0,6)} \end{array} \right)$$

$$+ \left(\begin{array}{c} 1 - \dfrac{|1-1|+|2-2|+|3-3|+|4-5|}{4} \cdot \\[3mm] \dfrac{0,7.[(0,1.0,2+0,3.0,3+0,4.0,5+0,5.0,6]}{(0,1^2+0,3^2+0,4^2+0,5^2)+(0,2^2+0,3^2+0,5^2+0,6^2)-(0,1.0,2+0,3.0,3+0,4.0,5+0,5.0,6)} \end{array} \right)$$

$$= \frac{3}{4} \frac{0,3.0,61}{0,51+0,74-0,61} + \frac{3}{4} \frac{0,7.0,61}{0,51+0,74-0,61}$$

$$= \frac{0,549}{2,56} + \frac{1,281}{2,56} \cong 0,714$$

Definition 3.9 Let $A = \left\langle (a_1,a_2,a_3,a_4); \eta_A^1, \eta_A^2, ..., \eta_A^p \right\rangle$, $B = \left\langle (b_1,b_2,b_3,b_4); \eta_B^1, \eta_B^2, ..., \eta_B^p \right\rangle$ be two TFMNs in the set of real numbers $\mathbb{R}$. Then; Cosine similarity measure between TFMN A and B denoted $C(A,B)$ is defined as;

$$C(A,B) = \frac{1}{p} \sum_{k=1}^{p} \left(\frac{\sum_{i=1}^4 a_i^k . b_i^k}{\sqrt{\sum_{i=1}^4 (a_i^k)^2} . \sqrt{\sum_{i=1}^4 (b_i^k)^2}} \cdot \frac{\min\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\min\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\min\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...}{\max\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\max\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\max\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...} \right)$$

Proposition 3.10 Let $C(A,B)$ be a Jaccard similarity measure between TFMN's A and B . Then we have,

i. $0 \le C(A,B) \le 1$

ii. $C(A,B) = C(B,A)$

iii. $C(A,B) = 1$ *for* $A = B$, i.e. $\eta_A^i = \eta_B^i$, i=1,2,...,p

Proof 3.11:

i. it is clear from Definition 3.9

ii. $C(A,B) =$

$$\frac{1}{p}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4}a_i^k.b_i^k}{\sqrt{\sum_{i=1}^{4}(a_i^k)^2}.\sqrt{\sum_{i=1}^{4}(b_i^k)^2}}\cdot\frac{\min\left(\left(\eta_A^1\right)_k,\left(\eta_B^1\right)_k\right)+\min\left(\left(\eta_A^2\right)_k,\left(\eta_B^2\right)_k\right)+\min\left(\left(\eta_A^3\right)_k,\left(\eta_B^3\right)_k\right)+...}{\max\left(\left(\eta_A^1\right)_k,\left(\eta_B^1\right)_k\right)+\max\left(\left(\eta_A^2\right)_k,\left(\eta_B^2\right)_k\right)+\max\left(\left(\eta_A^3\right)_k,\left(\eta_B^3\right)_k\right)+...}\right)$$

$$=\frac{1}{p}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4}b_i^k.a_i^k}{\sqrt{\sum_{i=1}^{4}(b_i^k)^2}.\sqrt{\sum_{i=1}^{4}(a_i^k)^2}}\cdot\frac{\min\left(\left(\eta_B^1\right)_k,\left(\eta_A^1\right)_k\right)+\min\left(\left(\eta_B^2\right)_k,\left(\eta_A^2\right)_k\right)+\min\left(\left(\eta_B^3\right)_k,\left(\eta_A^3\right)_k\right)+...}{\max\left(\left(\eta_B^1\right)_k,\left(\eta_A^1\right)_k\right)+\max\left(\left(\eta_B^2\right)_k,\left(\eta_A^2\right)_k\right)+\max\left(\left(\eta_B^3\right)_k,\left(\eta_A^3\right)_k\right)+...}\right)$$

$$= C(B,A).$$

iii.

$$C(A,B) =\frac{1}{p}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4}a_i^k.b_i^k}{\sqrt{\sum_{i=1}^{4}(a_i^k)^2}.\sqrt{\sum_{i=1}^{4}(b_i^k)^2}}\cdot\frac{\min\left(\left(\eta_A^1\right)_k,\left(\eta_B^1\right)_k\right)+\min\left(\left(\eta_A^2\right)_k,\left(\eta_B^2\right)_k\right)+\min\left(\left(\eta_A^3\right)_k,\left(\eta_B^3\right)_k\right)+...}{\max\left(\left(\eta_A^1\right)_k,\left(\eta_B^1\right)_k\right)+\max\left(\left(\eta_A^2\right)_k,\left(\eta_B^2\right)_k\right)+\max\left(\left(\eta_A^3\right)_k,\left(\eta_B^3\right)_k\right)+...}\right)$$

$$=\frac{1}{p}\sum_{k=1}^{p}\left(\frac{\dfrac{a_1^k.b_1^k+a_2^k.b_2^k+a_3^k.b_3^k+a_4^k.b_4^k}{\sqrt{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}.\sqrt{(b_1^k)^2+(b_2^k)^2+(b_3^k)^2+(b_4^k)^2}}\cdot}{\dfrac{\min\left(\left(\eta_A^1\right)_k,\left(\eta_B^1\right)_k\right)+\min\left(\left(\eta_A^2\right)_k,\left(\eta_B^2\right)_k\right)+\min\left(\left(\eta_A^3\right)_k,\left(\eta_B^3\right)_k\right)+...}{\max\left(\left(\eta_A^1\right)_k,\left(\eta_B^1\right)_k\right)+\max\left(\left(\eta_A^2\right)_k,\left(\eta_B^2\right)_k\right)+\max\left(\left(\eta_A^3\right)_k,\left(\eta_B^3\right)_k\right)+...}}\right)$$

$$=\frac{1}{p}\sum_{k=1}^{p}\left(\frac{\dfrac{a_1^k.a_1^k+a_2^k.a_2^k+a_3^k.a_3^k+a_4^k.a_4^k}{\sqrt{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}.\sqrt{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}}\cdot}{\dfrac{\min\left(\left(\eta_A^1\right)_k,\left(\eta_A^1\right)_k\right)+\min\left(\left(\eta_A^2\right)_k,\left(\eta_A^2\right)_k\right)+\min\left(\left(\eta_A^3\right)_k,\left(\eta_A^3\right)_k\right)+...}{\max\left(\left(\eta_A^1\right)_k,\left(\eta_A^1\right)_k\right)+\max\left(\left(\eta_A^2\right)_k,\left(\eta_A^2\right)_k\right)+\max\left(\left(\eta_A^3\right)_k,\left(\eta_A^3\right)_k\right)+...}}\right)$$

$$=\frac{1}{p}\sum_{k=1}^{p}\left(\frac{\dfrac{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}{\sqrt{((a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2)^2}}\cdot}{\dfrac{\min\left(\left(\eta_A^1\right)_k,\left(\eta_A^1\right)_k\right)+\min\left(\left(\eta_A^2\right)_k,\left(\eta_A^2\right)_k\right)+\min\left(\left(\eta_A^3\right)_k,\left(\eta_A^3\right)_k\right)+...}{\max\left(\left(\eta_A^1\right)_k,\left(\eta_A^1\right)_k\right)+\max\left(\left(\eta_A^2\right)_k,\left(\eta_A^2\right)_k\right)+\max\left(\left(\eta_A^3\right)_k,\left(\eta_A^3\right)_k\right)+...}}\right)$$

$$= \frac{1}{p}\sum_{k=1}^{p}\left(\frac{\dfrac{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}\cdot}{\dfrac{\min\left((\eta_A^1)_k,(\eta_A^1)_k\right)+\min\left((\eta_A^2)_k,(\eta_A^2)_k\right)+\min\left((\eta_A^3)_k,(\eta_A^3)_k\right)+...}{\max\left((\eta_A^1)_k,(\eta_A^1)_k\right)+\max\left((\eta_A^2)_k,(\eta_A^2)_k\right)+\max\left((\eta_A^3)_k,(\eta_A^3)_k\right)+...}}\right)$$

$$= \frac{1}{p}\sum_{k=1}^{p}\left(1.\frac{(\eta_A^1)_k+(\eta_A^2)_k+(\eta_A^3)_k+...}{(\eta_A^1)_k+(\eta_A^1)_k+(\eta_A^3)_k+...}\right)$$

$$=1.$$

Example 3.12 Let $A=\langle(0.3,0.2,0.4,0.5);0.1,0.2,0.3,0.4\rangle$ and

$B=\langle(0.1,0.3,0.5,0.6);0.2,0.4,0.6,0.8\rangle$ between TFMN's . Then $C(A,B)$ be a cosine similarity measure;

$$C(\text{A,B})=\frac{1}{p}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4}a_i^k.b_i^k}{\sqrt{\sum_{i=1}^{4}(a_i^k)^2}.\sqrt{\sum_{i=1}^{4}(b_i^k)^2}}\cdot\frac{\min\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\min\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\min\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...}{\max\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\max\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\max\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...}\right)$$

$$=\frac{(0,3.0,1+0,2.0,3+0,4.0,5+0,5.0,6)}{\sqrt{(0,3)^2+(0,2)^2+(0,4)^2+(0,5)^2}.\sqrt{(0,1)^2+(0,3)^2+(0,5)^2+(0,6)^2}}\cdot\frac{(0,1+0,2+0,3+0,4)}{(0,2+0,4+0,6+0,8)}$$

$$\cong 0,47642$$

Definition 3.13 Let $A=\langle(a_1,a_2,a_3,a_4);\eta_A^1,\eta_A^2,...,\eta_A^p\rangle$, $B=\langle(b_1,b_2,b_3,b_4);\eta_B^1,\eta_B^2,...,\eta_B^p\rangle$ be two TFMNs in the set of real numbers $\mathbb{R}$ and $w_i\in[0,1]$ be the weight of each element for $i=(1,2,...,n)$ such that $\sum_{i=1}^{n}w_i=1$. Then; Cosine similarity measure between TFMN A and B denoted $C_w(\text{A,B})$ is defined as;

$$C_w(\text{A,B})=\frac{1}{p}\sum_{i=1}^{n}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4}a_i^k.b_i^k}{\sqrt{\sum_{i=1}^{4}(a_i^k)^2}.\sqrt{\sum_{i=1}^{4}(b_i^k)^2}}\cdot\frac{w_i.\left[\min\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\min\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\min\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...\right]}{\max\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\max\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\max\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...}\right)$$

Proposition 3.14 Let $C_w(A,B)$ be a weighted Cosine similarity measure between TFMN's A and B, $w_i \in [0,1]$ be the weight of each element for $i = (1,2,...,n)$ such that $\sum_{i=1}^{n} w_i = 1$ Then weighted Cosine vector similarity measure between TFMN's A and B;

i. $0 \le C_w(A,B) \le 1$

ii. $C_w(A,B) = C_w(B,A)$

iii. $C_w(A,B) = 1$ *for* $A = B$ i.e. $(\eta_A^1 = \eta_B^1, \eta_A^2 = \eta_B^2,..., \eta_A^p = \eta_B^p)$

Proof 3.15:

i. it is clear from Definition 3.13

ii.

$$C_w(A,B) = \frac{1}{p}\sum_{i=1}^{n}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4} a_i^k . b_i^k}{\sqrt{\sum_{i=1}^{4}(a_i^k)^2} . \sqrt{\sum_{i=1}^{4}(b_i^k)^2}} \cdot \frac{w_i \cdot \left[\min\left(\left(\eta_A^1\right)_k,\left(\eta_B^1\right)_k\right) + \min\left(\left(\eta_A^2\right)_k,\left(\eta_B^2\right)_k\right) + \min\left(\left(\eta_A^3\right)_k,\left(\eta_B^3\right)_k\right) + ... \right]}{\max\left(\left(\eta_A^1\right)_k,\left(\eta_B^1\right)_k\right) + \max\left(\left(\eta_A^2\right)_k,\left(\eta_B^2\right)_k\right) + \max\left(\left(\eta_A^3\right)_k,\left(\eta_B^3\right)_k\right) + ...} \right)$$

$$= \frac{1}{p}\sum_{i=1}^{n}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4} a_i^k . b_i^k}{\sqrt{\sum_{i=1}^{4}(a_i^k)^2} . \sqrt{\sum_{i=1}^{4}(b_i^k)^2}} \cdot \frac{w_i \cdot \left[\min\left(\left(\eta_A^1\right)_k,\left(\eta_B^1\right)_k\right) + \min\left(\left(\eta_A^2\right)_k,\left(\eta_B^2\right)_k\right) + \min\left(\left(\eta_A^3\right)_k,\left(\eta_B^3\right)_k\right) + ... \right]}{\max\left(\left(\eta_A^1\right)_k,\left(\eta_B^1\right)_k\right) + \max\left(\left(\eta_A^2\right)_k,\left(\eta_B^2\right)_k\right) + \max\left(\left(\eta_A^3\right)_k,\left(\eta_B^3\right)_k\right) + ...} \right)$$

$$= \frac{1}{p}\sum_{i=1}^{n}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4} b_i^k . a_i^k}{\sqrt{\sum_{i=1}^{4}(b_i^k)^2} . \sqrt{\sum_{i=1}^{4}(a_i^k)^2}} \cdot \frac{w_i \cdot \left[\min\left(\left(\eta_B^1\right)_k,\left(\eta_A^1\right)_k\right) + \min\left(\left(\eta_B^2\right)_k,\left(\eta_A^2\right)_k\right) + \min\left(\left(\eta_B^3\right)_k,\left(\eta_A^3\right)_k\right) + ... \right]}{\max\left(\left(\eta_B^1\right)_k,\left(\eta_A^1\right)_k\right) + \max\left(\left(\eta_B^2\right)_k,\left(\eta_A^2\right)_k\right) + \max\left(\left(\eta_B^3\right)_k,\left(\eta_A^3\right)_k\right) + ...} \right)$$

$$= C_w(B,A).$$

iii.

$$C_w(A,B) = \frac{1}{p}\sum_{i=1}^{n}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4} a_i^k . b_i^k}{\sqrt{\sum_{i=1}^{4}(a_i^k)^2} . \sqrt{\sum_{i=1}^{4}(b_i^k)^2}} \cdot \frac{w_i \cdot \left[\min\left(\left(\eta_A^1\right)_k,\left(\eta_B^1\right)_k\right) + \min\left(\left(\eta_A^2\right)_k,\left(\eta_B^2\right)_k\right) + \min\left(\left(\eta_A^3\right)_k,\left(\eta_B^3\right)_k\right) + ... \right]}{\max\left(\left(\eta_A^1\right)_k,\left(\eta_B^1\right)_k\right) + \max\left(\left(\eta_A^2\right)_k,\left(\eta_B^2\right)_k\right) + \max\left(\left(\eta_A^3\right)_k,\left(\eta_B^3\right)_k\right) + ...} \right)$$

$$= \frac{1}{p} \sum_{i=1}^{n} \sum_{k=1}^{p} \left(\frac{\dfrac{a_1^k . b_1^k + a_2^k . b_2^k + a_3^k . b_3^k + a_4^k . b_4^k}{\sqrt{(a_1^k)^2 + (a_2^k)^2 + (a_3^k)^2 + (a_4^k)^2} . \sqrt{(b_1^k)^2 + (b_2^k)^2 + (b_3^k)^2 + (b_4^k)^2}} \cdot}{w_i \cdot \left[\min\left((\eta_A^1)_k, (\eta_B^1)_k\right) + \min\left((\eta_A^2)_k, (\eta_B^2)_k\right) + \min\left((\eta_A^3)_k, (\eta_B^3)_k\right) + \dots \right] } \over { \max\left((\eta_A^1)_k, (\eta_B^1)_k\right) + \max\left((\eta_A^2)_k, (\eta_B^2)_k\right) + \max\left((\eta_A^3)_k, (\eta_B^3)_k\right) + \dots } \right)$$

$$= \frac{1}{p} \sum_{i=1}^{n} \sum_{k=1}^{p} \left(\frac{\dfrac{a_1^k . a_1^k + a_2^k . a_2^k + a_3^k . a_3^k + a_4^k . a_4^k}{\sqrt{(a_1^k)^2 + (a_2^k)^2 + (a_3^k)^2 + (a_4^k)^2} . \sqrt{(a_1^k)^2 + (a_2^k)^2 + (a_3^k)^2 + (a_4^k)^2}} \cdot}{w_i \cdot \left[\min\left((\eta_A^1)_k, (\eta_A^1)_k\right) + \min\left((\eta_A^2)_k, (\eta_A^2)_k\right) + \min\left((\eta_A^3)_k, (\eta_A^3)_k\right) + \dots \right]} \over {\max\left((\eta_A^1)_k, (\eta_A^1)_k\right) + \max\left((\eta_A^2)_k, (\eta_A^2)_k\right) + \max\left((\eta_A^3)_k, (\eta_A^3)_k\right) + \dots} \right)$$

$$= \frac{1}{p} \sum_{i=1}^{n} \sum_{k=1}^{p} \left(\frac{\dfrac{(a_1^k)^2 + (a_2^k)^2 + (a_3^k)^2 + (a_4^k)^2}{\sqrt{((a_1^k)^2 + (a_2^k)^2 + (a_3^k)^2 + (a_4^k)^2)^2}} \cdot}{w_i \cdot \left[\min\left((\eta_A^1)_k, (\eta_A^1)_k\right) + \min\left((\eta_A^2)_k, (\eta_A^2)_k\right) + \min\left((\eta_A^3)_k, (\eta_A^3)_k\right) + \dots \right]} \over {\max\left((\eta_A^1)_k, (\eta_A^1)_k\right) + \max\left((\eta_A^2)_k, (\eta_A^2)_k\right) + \max\left((\eta_A^3)_k, (\eta_A^3)_k\right) + \dots} \right)$$

$$= \frac{1}{p} \sum_{i=1}^{n} \sum_{k=1}^{p} \left(\frac{\dfrac{(a_1^k)^2 + (a_2^k)^2 + (a_3^k)^2 + (a_4^k)^2}{(a_1^k)^2 + (a_2^k)^2 + (a_3^k)^2 + (a_4^k)^2} \cdot}{w_i \cdot \left[\min\left((\eta_A^1)_k, (\eta_A^1)_k\right) + \min\left((\eta_A^2)_k, (\eta_A^2)_k\right) + \min\left((\eta_A^3)_k, (\eta_A^3)_k\right) + \dots \right]} \over {\max\left((\eta_A^1)_k, (\eta_A^1)_k\right) + \max\left((\eta_A^2)_k, (\eta_A^2)_k\right) + \max\left((\eta_A^3)_k, (\eta_A^3)_k\right) + \dots} \right)$$

$$= \sum_{i=1}^{n} w_i$$

$$= 1.$$

Example 3.16 Let $A = \langle (0.1, 0.2, 0.3, 0.4); 0.2, 0.3, 0.5, 0.6 \rangle$ and

$B = \langle (0.1, 0.2, 0.4, 0.5); 0.1, 0.4, 0.5, 0.7 \rangle$ be two TFMNs in the set of real numbers $\mathbb{R}$ and w_i

be the weight of each element for $i = (1, 2)$ $w_1 = 0.2, w_2 = 0.8$ such that $\sum_{i=1}^{n} w_i = 1$. Then;

Cosine similarity measure between TFMN A and B is;

$$C_w(\text{A, B}) = \frac{1}{p}\sum_{i=1}^{n}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4} a_i^k . b_i^k}{\sqrt{\sum_{i=1}^{4}(a_i^k)^2}.\sqrt{\sum_{i=1}^{4}(b_i^k)^2}} . \frac{w_i.\left[\min\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\min\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\min\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...\right]}{\max\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\max\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\max\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...}\right)$$

$$=\frac{0,2.(0,1.0,1+0,2.0,2+0,3.0,4+0,4.0,5)}{\sqrt{(0,1)^2+(0,2)^2+(0,3)^2+(0,4)^2}.\sqrt{(0,1)^2+(0,2)^2+(0,4)^2+(0,5)^2}} . \frac{(0,1+0,3+0,5+0,6)}{(0,2+0,4+0,5+0,7)}$$

$$+\frac{0,8.(0,1.0,1+0,2.0,2+0,3.0,4+0,4.0,5)}{\sqrt{(0,1)^2+(0,2)^2+(0,3)^2+(0,4)^2}.\sqrt{(0,1)^2+(0,2)^2+(0,4)^2+(0,5)^2}} . \frac{(0,1+0,3+0,5+0,6)}{(0,2+0,4+0,5+0,7)}$$

$$\cong 0,830$$

4. HYBRID SIMILARITY MEASURE FOR TRAPEZOIDAL FUZZY MULTI NUMBERS

In this section, we introduced some similarity measures for TFMNs including Hybrid similarity measure.

Definition 4.1: Let $A=\left\langle(a_1,a_2,a_3,a_4);\eta_A^1,\eta_A^2,...,\eta_A^p\right\rangle$, $B=\left\langle(b_1,b_2,b_3,b_4);\eta_B^1,\eta_B^2,...,\eta_B^p\right\rangle$ be two TFMNs in the set of real numbers $\mathbb{R}$. Then, hybrid vector similarity measure between TFMN A and B, denoted $HybV(A,\ B)$, is defined as;

$$HybV(A,\ B)=\lambda.\left(\frac{1}{p}\sum_{k=1}^{p}\left(\left(1-\frac{\sum_{j=1}^{4}\left|a_j^k-b_j^k\right|}{4}\right).\frac{\sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}{\sum_{i=1}^{p}(\eta_A^i)_k^2+\sum_{i=1}^{p}(\eta_B^i)_k^2-\sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}\right)\right)+$$

$$(1-\lambda).\left(\frac{1}{p}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4} a_i^k . b_i^k}{\sqrt{\sum_{i=1}^{4}(a_i^k)^2}.\sqrt{\sum_{i=1}^{4}(b_i^k)^2}} . \frac{\min\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\min\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\min\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...}{\max\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\max\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\max\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...}\right)\right)$$

Example 4.2 Let $HybV(A,\ B)$ be a hybrid vector similarity measure between TFMN's $A=\left\langle(2,4,5,7);0.2,0.4,0.6,0.8\right\rangle$ and $B=\left\langle(1,4,5,9);0.5,0.6,0.7,0.8\right\rangle$. Then $HybV(A,\ B)$ be a hybrid vector similarity measure;

$$HybV\left(A,\ B\right)=\lambda.\left(\frac{1}{p}\sum_{k=1}^{p}\left(\left(1-\frac{\sum_{j=1}^{4}\left|a_{j}^{k}-b_{j}^{k}\right|}{4}\right).\frac{\sum_{i=1}^{p}(\eta_{A}^{i})_{k}.(\eta_{B}^{i})_{k}}{\sum_{i=1}^{p}(\eta_{A}^{i})_{k}^{2}+\sum_{i=1}^{p}(\eta_{B}^{i})_{k}^{2}-\sum_{i=1}^{p}(\eta_{A}^{i})_{k}.(\eta_{B}^{i})_{k}}\right)\right)+$$

$$(1-\lambda).\left(\frac{1}{p}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4}a_{i}^{k}.b_{i}^{k}}{\sqrt{\sum_{i=1}^{4}(a_{i}^{k})^{2}}.\sqrt{\sum_{i=1}^{4}(b_{i}^{k})^{2}}}.\frac{\min\left(\left(\eta_{A}^{1}\right)_{k},\left(\eta_{B}^{1}\right)_{k}\right)+\min\left(\left(\eta_{A}^{2}\right)_{k},\left(\eta_{B}^{2}\right)_{k}\right)+\min\left(\left(\eta_{A}^{3}\right)_{k},\left(\eta_{B}^{3}\right)_{k}\right)+...}{\max\left(\left(\eta_{A}^{1}\right)_{k},\left(\eta_{B}^{1}\right)_{k}\right)+\max\left(\left(\eta_{A}^{2}\right)_{k},\left(\eta_{B}^{2}\right)_{k}\right)+\max\left(\left(\eta_{A}^{3}\right)_{k},\left(\eta_{B}^{3}\right)_{k}\right)+...}\right)\right)$$

$$=0,4.\left(\frac{1-\frac{\left|2-1\right|+\left|4-4\right|+\left|5-5\right|+\left|7-9\right|}{4}.}{\left[(0,2)^{2}+(0,4)^{2}+(0,6)^{2}+(0,8)^{2}\right]+\left[(0,5)^{2}+(0,6)^{2}+(0,7)^{2}+(0,8)^{2}\right]-\left[(0,2.0,5+0,4.0,6+0,6.0,7+0,8.0,8)\right]}\right)$$

$$+(1-0,4).\left(\frac{2.1+4.4+5.5+7.9}{\sqrt{(2^{2}+4^{2}+5^{2}+7^{2})}\sqrt{(1^{2}+4^{2}+5^{2}+9^{2})}}.\frac{0,2+0,4+0,6+0,8}{0,5+0,6+0,7+0,8}\right)$$

$$=0,4.\left(\frac{1}{4}.\frac{(1,4)}{1,54}\right)+(0,6).\left(\frac{106}{\sqrt{94}\sqrt{123}}.\frac{2}{2,6}\right)$$

$$\cong0,3639$$

Proposition 4.3 Let $HybV\left(A,\ B\right)$ be a Hybrid similarity measure between TFMN's A and B. Then we have,

i. $0\leq HybV\left(A,\ B\right)\leq1$

ii. $HybV\left(A,B\right)=HybV\left(B,A\right)$

iii. $HybV\left(A,\ B\right)=1$ *for* $A=B$, i.e. $\eta_{A}^{i}=\eta_{B}^{i}$, (i=1,2,...,p)

Proof 4.4:

i. it is clear from Definition 4.1

ii.

$$HybV\left(A,\ B\right) = \lambda.\left(\frac{1}{p}\sum_{k=1}^{p}\left(\left(1 - \frac{\sum_{j=1}^{4}\left|a_j^k - b_j^k\right|}{4}\right)\cdot\frac{\sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}{\sum_{i=1}^{p}(\eta_A^i)_k^{\,2} + \sum_{i=1}^{p}(\eta_B^i)_k^{\,2} - \sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}\right)\right) +$$

$$(1-\lambda).\left(\frac{1}{p}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4}a_i^k.b_i^k}{\sqrt{\sum_{i=1}^{4}(a_i^k)^2}.\sqrt{\sum_{i=1}^{4}(b_i^k)^2}}\cdot\frac{\min\left((\eta_A^1)_k,(\eta_B^1)_k\right) + \min\left((\eta_A^2)_k,(\eta_B^2)_k\right) + \min\left((\eta_A^3)_k,(\eta_B^3)_k\right) + \ldots}{\max\left((\eta_A^1)_k,(\eta_B^1)_k\right) + \max\left((\eta_A^2)_k,(\eta_B^2)_k\right) + \max\left((\eta_A^3)_k,(\eta_B^3)_k\right) + \ldots}\right)\right)$$

$$= \lambda.\left(\frac{1}{p}\sum_{k=1}^{p}\left(\left(1 - \frac{\sum_{j=1}^{4}\left|b_j^k - a_j^k\right|}{4}\right)\cdot\frac{\sum_{i=1}^{p}(\eta_B^i)_k.(\eta_A^i)_k}{\sum_{i=1}^{p}(\eta_B^i)_k^{\,2} + \sum_{i=1}^{p}(\eta_A^i)_k^{\,2} - \sum_{i=1}^{p}(\eta_B^i)_k.(\eta_A^i)_k}\right)\right) +$$

$$(1-\lambda).\left(\frac{1}{p}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4}b_i^k.a_i^k}{\sqrt{\sum_{i=1}^{4}(b_i^k)^2}.\sqrt{\sum_{i=1}^{4}(a_i^k)^2}}\cdot\frac{\min\left((\eta_B^1)_k,(\eta_A^1)_k\right) + \min\left((\eta_B^2)_k,(\eta_A^2)_k\right) + \min\left((\eta_B^3)_k,(\eta_A^3)_k\right) + \ldots}{\max\left((\eta_B^1)_k,(\eta_A^1)_k\right) + \max\left((\eta_B^2)_k,(\eta_A^2)_k\right) + \max\left((\eta_B^3)_k,(\eta_A^3)_k\right) + \ldots}\right)\right)$$

$$= HybV\left(B,A\right).$$

iii.

$$HybV\left(A,\ B\right) = \lambda.\left(\frac{1}{p}\sum_{k=1}^{p}\left(\left(1 - \frac{\sum_{j=1}^{4}\left|a_j^k - b_j^k\right|}{4}\right)\cdot\frac{\sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}{\sum_{i=1}^{p}(\eta_A^i)_k^{\,2} + \sum_{i=1}^{p}(\eta_B^i)_k^{\,2} - \sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}\right)\right) +$$

$$(1-\lambda).\left(\frac{1}{p}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4}a_i^k.b_i^k}{\sqrt{\sum_{i=1}^{4}(a_i^k)^2}.\sqrt{\sum_{i=1}^{4}(b_i^k)^2}}\cdot\frac{\min\left((\eta_A^1)_k,(\eta_B^1)_k\right) + \min\left((\eta_A^2)_k,(\eta_B^2)_k\right) + \min\left((\eta_A^3)_k,(\eta_B^3)_k\right) + \ldots}{\max\left((\eta_A^1)_k,(\eta_B^1)_k\right) + \max\left((\eta_A^2)_k,(\eta_B^2)_k\right) + \max\left((\eta_A^3)_k,(\eta_B^3)_k\right) + \ldots}\right)\right)$$

$$= \lambda.\left(\frac{1}{p}\sum_{k=1}^{p}\left(\left(1 - \frac{\left|a_1^k - b_1^k\right| + \left|a_2^k - b_2^k\right| + \left|a_3^k - b_3^k\right| + \left|a_4^k - b_4^k\right|}{4}\right)\cdot\frac{(\eta_A^1)_k.(\eta_B^1)_k + (\eta_A^2)_k.(\eta_B^2)_k + \ldots + (\eta_A^p)_k.(\eta_B^p)_k}{(\eta_A^1)_k^2 + \ldots + (\eta_A^p)_k^2 + (\eta_B^1)_k^2 + \ldots + (\eta_B^p)_k^2 - ((\eta_A^1)_k.(\eta_B^1)_k + \ldots(\eta_A^p)_k.(\eta_B^p)_k)}\right)\right) +$$

$$(1-\lambda).\left(\frac{1}{p}\sum_{k=1}^{p}\left(\frac{a_1^k.b_1^k+a_2^k.b_2^k+a_3^k.b_3^k+a_4^k.b_4^k}{\sqrt{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}.\sqrt{(b_1^k)^2+(b_2^k)^2+(b_3^k)^2+(b_4^k)^2}}\right).\frac{\min\left(\left(\eta_A^1\right)_k,\left(\eta_B^1\right)_k\right)+\min\left(\left(\eta_A^2\right)_k,\left(\eta_B^2\right)_k\right)+\min\left(\left(\eta_A^3\right)_k,\left(\eta_B^3\right)_k\right)+...}{\max\left(\left(\eta_A^1\right)_k,\left(\eta_B^1\right)_k\right)+\max\left(\left(\eta_A^2\right)_k,\left(\eta_B^2\right)_k\right)+\max\left(\left(\eta_A^3\right)_k,\left(\eta_B^3\right)_k\right)+...}\right)$$

$$=\lambda.\left(\frac{1}{p}\sum_{k=1}^{p}\left(\left(1-\frac{\left|a_1^k-a_1^k\right|+\left|a_2^k-a_2^k\right|+\left|a_3^k-a_3^k\right|+\left|a_4^k-a_4^k\right|}{4}\right).\frac{(\eta_A^1)_k.(\eta_A^1)_k+(\eta_A^2)_k.(\eta_A^2)_k+....+(\eta_A^p)_k.(\eta_A^p)_k}{(\eta_A^1)_k^2+...+(\eta_A^p)_k^2+(\eta_A^1)_k^2+...+(\eta_A^p)_k^2-((\eta_A^1)_k^2+...(\eta_A^p)_k^2)}\right)\right)+$$

$$(1-\lambda).\left(\frac{1}{p}\sum_{k=1}^{p}\left(\frac{a_1^k.a_1^k+a_2^k.a_2^k+a_3^k.a_3^k+a_4^k.a_4^k}{\sqrt{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}.\sqrt{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}}\right).\frac{\min\left(\left(\eta_A^1\right)_k,\left(\eta_A^1\right)_k\right)+\min\left(\left(\eta_A^2\right)_k,\left(\eta_A^2\right)_k\right)+\min\left(\left(\eta_A^3\right)_k,\left(\eta_A^3\right)_k\right)+...}{\max\left(\left(\eta_A^1\right)_k,\left(\eta_A^1\right)_k\right)+\max\left(\left(\eta_A^2\right)_k,\left(\eta_A^2\right)_k\right)+\max\left(\left(\eta_A^3\right)_k,\left(\eta_A^3\right)_k\right)+...}\right)$$

$$=\lambda.\left(\frac{1}{p}\sum_{k=1}^{p}\left((1-0).\frac{(\eta_A^1)_k.(\eta_A^1)_k+(\eta_A^2)_k.(\eta_A^2)_k+....+(\eta_A^p)_k.(\eta_A^p)_k}{(\eta_A^1)_k^2+...+(\eta_A^p)_k^2+(\eta_A^1)_k^2+...+(\eta_A^p)_k^2-((\eta_A^1)_k^2+...(\eta_A^p)_k^2)}\right)\right)+$$

$$(1-\lambda).\left(\frac{1}{p}\sum_{k=1}^{p}\left(\frac{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}.\frac{\min\left(\left(\eta_A^1\right)_k,\left(\eta_A^1\right)_k\right)+\min\left(\left(\eta_A^2\right)_k,\left(\eta_A^2\right)_k\right)+\min\left(\left(\eta_A^3\right)_k,\left(\eta_A^3\right)_k\right)+...}{\max\left(\left(\eta_A^1\right)_k,\left(\eta_A^1\right)_k\right)+\max\left(\left(\eta_A^2\right)_k,\left(\eta_A^2\right)_k\right)+\max\left(\left(\eta_A^3\right)_k,\left(\eta_A^3\right)_k\right)+...}\right)\right)$$

$$=\lambda.\left(\frac{1}{p}\sum_{k=1}^{p}\left(\frac{(\eta_A^1)_k^2+(\eta_A^2)_k^2+...+(\eta_A^p)_k^2}{(\eta_A^1)_k^2+(\eta_A^2)_k^2+...+(\eta_A^p)_k^2}\right)\right)+(1-\lambda).\left(\frac{1}{p}\sum_{k=1}^{p}\left(\begin{array}{c}\frac{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}.\\ \frac{\min\left(\left(\eta_A^1\right)_k,\left(\eta_A^1\right)_k\right)+\min\left(\left(\eta_A^2\right)_k,\left(\eta_A^2\right)_k\right)+\min\left(\left(\eta_A^3\right)_k,\left(\eta_A^3\right)_k\right)+...}{\max\left(\left(\eta_A^1\right)_k,\left(\eta_A^1\right)_k\right)+\max\left(\left(\eta_A^2\right)_k,\left(\eta_A^2\right)_k\right)+\max\left(\left(\eta_A^3\right)_k,\left(\eta_A^3\right)_k\right)+...}\end{array}\right)\right)$$

$$=\lambda.(1)+(1-\lambda).(1)$$

$$=1.$$

Definition 4.5 Let $A=\left\langle(a_1,a_2,a_3,a_4);\eta_A^1,\eta_A^2,...,\eta_A^p\right\rangle$, $B=\left\langle(b_1,b_2,b_3,b_4);\eta_B^1,\eta_B^2,...,\eta_B^p\right\rangle$ be

two TFMNs in the set of real numbers $\mathbb{R}$ and $w_q\in[0,1]$ be the weight number for

$q = (1, 2, ..., n)$ such that $\sum_{q=1}^{n} w_q = 1$. Then; hybrid vector similarity measure between TFMN A

and B denoted $HybV_w(A, B)$ is defined as;

$$HybV_w(A, B) = \lambda.\left(\frac{1}{p}\sum_{q=1}^{n}\sum_{k=1}^{p}\left(\left(1 - \frac{\sum_{j=1}^{4}|a_j^k - b_j^k|}{4}\right).\frac{w_q.\sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}{\sum_{i=1}^{p}(\eta_A^i)_k^2 + \sum_{i=1}^{p}(\eta_B^i)_k^2 - \sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}\right)\right) +$$

$$(1-\lambda).\left(\frac{1}{p}\sum_{q=1}^{n}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4}a_i^k.b_i^k}{\sqrt{\sum_{i=1}^{4}(a_i^k)^2}.\sqrt{\sum_{i=1}^{4}(b_i^k)^2}}.\frac{w_q.\left[\min\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\min\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\min\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...\right]}{\max\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\max\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\max\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...}\right)\right)$$

Example 4.6 Let $A = \langle(0.2, 0.3, 0.5, 0.6); 0.1, 0.4, 0.5, 0.6\rangle$,

$B = \langle(0.1, 0.3, 0.4, 0.6); 0.2, 0.4, 0.6, 0.8\rangle$ be two TFMNs in the set of real numbers $\mathbb{R}$ and

w_q be the weight number for $q = (1, 2)$ $w_1 = 0.4, w_2 = 0.6$ such that $\sum_{i=1}^{n} w_q = 1$. Then; hybrid

vector similarity measure between TFMN A and B is;

$$HybV_w(A, B) = \lambda.\left(\frac{1}{p}\sum_{q=1}^{n}\sum_{k=1}^{p}\left(\left(1 - \frac{\sum_{j=1}^{4}|a_j^k - b_j^k|}{4}\right).\frac{w_q.\sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}{\sum_{i=1}^{p}(\eta_A^i)_k^2 + \sum_{i=1}^{p}(\eta_B^i)_k^2 - \sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}\right)\right) +$$

$$(1-\lambda).\left(\frac{1}{p}\sum_{q=1}^{n}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4}a_i^k.b_i^k}{\sqrt{\sum_{i=1}^{4}(a_i^k)^2}.\sqrt{\sum_{i=1}^{4}(b_i^k)^2}}.\frac{w_q.\left[\min\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\min\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\min\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...\right]}{\max\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\max\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\max\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...}\right)\right)$$

$$= 0,3.\left(1 - \frac{|0,2-0,1|+|0,3-0,3|+|0,5-0,4|+|0,6-0,6|}{4}.\right.$$
$$\left.\frac{0,4.(0,1.0,2+0,4.0,4+0,5.0,6+0,6.0,8)}{[(0,1)^2+(0,4)^2+(0,5)^2+(0,6)^2]+[(0,2)^2+(0,4)^2+(0,6)^2+(0,8)^2]-[(0,1.0,2+0,4.0,4+0,5.0,6+0,6.0,8)]}\right)$$

$$+0,3.\left(\begin{array}{c} 1-\dfrac{|0,2-0,1|+|0,3-0,3|+|0,5-0,4|+|0,6-0,6|}{4}. \\[4mm] \dfrac{0,6.(0,1.0,2+0,4.0,4+0,5.0,6+0,6.0,8)}{[(0,1)^2+(0,4)^2+(0,5)^2+(0,6)^2]+[(0,2)^2+(0,4)^2+(0,6)^2+(0,8)^2]-[(0,1.0,2+0,4.0,4+0,5.0,6+0,6.0,8)]} \end{array}\right)$$

$$+(0,7).\left(\dfrac{0,2.0,1+0,3.0,3+0,5.0,4+0,6.0,6}{\sqrt{(0,2)^2+(0,3)^2+(0,5)^2+(0,6)^2}+\sqrt{(0,1)^2+(0,3)^2+(0,4)^2+(0,6)^2}}.0,4.\dfrac{0,1+0,4+0,5+0,6}{0,2+0,4+0,6+0,8}\right)$$

$$+(0,7).\left(\dfrac{0,2.0,1+0,3.0,3+0,5.0,4+0,6.0,6}{\sqrt{(0,2)^2+(0,3)^2+(0,5)^2+(0,6)^2}+\sqrt{(0,1)^2+(0,3)^2+(0,4)^2+(0,6)^2}}.0,6.\dfrac{0,1+0,4+0,5+0,6}{0,2+0,4+0,6+0,8}\right)$$

$$\cong 0,26823+0,55392\cong 0,82215$$

Proposition 4.7 Let $HybV_w\left(A,\ B\right)$ be a hybrid vector similarity measure between TFMN's

A and B. $w_q \in [0,1]$ be the weight number, such that $\sum_{q=1}^{n} w_q = 1$. Then we have,

i. $0 \le HybV_w\left(A,\ B\right) \le 1$

ii. $HybV_w\left(A,B\right) = HybV_w\left(B,A\right)$

iii. $HybV_w\left(A,\ B\right) = 1$ *for* $A = B$, i.e. $\eta_A^i = \eta_B^i$, (i=1,2,...,p)

Proof 4.8:

i. it is clear from Definition 4.5

ii.

$$HybV_w\left(A,\ B\right) = \lambda.\left(\dfrac{1}{p}\sum_{q=1}^{n}\sum_{k=1}^{p}\left(\left(1-\dfrac{\sum_{j=1}^{4}|a_j^k-b_j^k|}{4}\right).\dfrac{w_q.\sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}{\sum_{i=1}^{p}(\eta_A^i)_k^{\,2}+\sum_{i=1}^{p}(\eta_B^i)_k^{\,2}-\sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}\right)\right)$$

$$+(1-\lambda).\left(\frac{1}{p}\sum_{q=1}^{n}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4}a_i^k.b_i^k}{\sqrt{\sum_{i=1}^{4}(a_i^k)^2}.\sqrt{\sum_{i=1}^{4}(b_i^k)^2}}.\frac{w_q.\left[\min\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\min\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\min\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...\right]}{\max\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\max\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\max\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...}\right)\right)$$

$$=\lambda.\left(\frac{1}{p}\sum_{q=1}^{n}\sum_{k=1}^{p}\left(\left(1-\frac{\sum_{j=1}^{4}\left|b_j^k-a_j^k\right|}{4}\right).\frac{w_q.\sum_{i=1}^{p}(\eta_B^i)_k.(\eta_A^i)_k}{\sum_{i=1}^{p}(\eta_B^i)_k^2+\sum_{i=1}^{p}(\eta_A^i)_k^2-\sum_{i=1}^{p}(\eta_B^i)_k.(\eta_A^i)_k}\right)\right)$$

$$+(1-\lambda).\left(\frac{1}{p}\sum_{q=1}^{n}\sum_{i=1}^{p}\left(\frac{\sum_{i=1}^{4}b_i^k.a_i^k}{\sqrt{\sum_{i=1}^{4}(b_i^k)^2}.\sqrt{\sum_{i=1}^{4}(a_i^k)^2}}.\frac{w_q.\left[\min\left((\eta_B^1)_k,(\eta_A^1)_k\right)+\min\left((\eta_B^2)_k,(\eta_A^2)_k\right)+\min\left((\eta_B^3)_k,(\eta_A^3)_k\right)+...\right]}{\max\left((\eta_B^1)_k,(\eta_A^1)_k\right)+\max\left((\eta_B^2)_k,(\eta_A^2)_k\right)+\max\left((\eta_B^3)_k,(\eta_A^3)_k\right)+...}\right)\right)$$

$$=HybV_w(B,A).$$

iii.

$$HybV_w(A,\ B)=\lambda.\left(\frac{1}{p}\sum_{q=1}^{n}\sum_{k=1}^{p}\left(\left(1-\frac{\sum_{j=1}^{4}\left|a_j^k-b_j^k\right|}{4}\right).\frac{w_q.\sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}{\sum_{i=1}^{p}(\eta_A^i)_k^2+\sum_{i=1}^{p}(\eta_B^i)_k^2-\sum_{i=1}^{p}(\eta_A^i)_k.(\eta_B^i)_k}\right)\right)$$

$$+(1-\lambda).\left(\frac{1}{p}\sum_{q=1}^{n}\sum_{k=1}^{p}\left(\frac{\sum_{i=1}^{4}a_i^k.b_i^k}{\sqrt{\sum_{i=1}^{4}(a_i^k)^2}.\sqrt{\sum_{i=1}^{4}(b_i^k)^2}}.\frac{w_q.\left[\min\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\min\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\min\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...\right]}{\max\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\max\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\max\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...}\right)\right)$$

$$=\lambda.\left(\frac{1}{p}\sum_{q=1}^{n}\sum_{k=1}^{p}\left(\left(1-\frac{\left|a_1^k-b_1^k\right|+\left|a_2^k-b_2^k\right|+\left|a_3^k-b_3^k\right|+\left|a_4^k-b_4^k\right|}{4}\right).\frac{w_q.\left[(\eta_A^1)_k.(\eta_B^1)_k+(\eta_A^2)_k.(\eta_B^2)_k+....+(\eta_A^p)_k.(\eta_B^p)_k\right]}{(\eta_A^1)_k^2+...+(\eta_A^p)_k^2+(\eta_B^1)_k^2+...+(\eta_B^p)_k^2-((\eta_A^1)_k.(\eta_B^1)_k+...(\eta_A^p)_k.(\eta_B^p)_k)}\right)\right)$$

$$+(1-\lambda).\left(\frac{1}{p}\sum_{q=1}^{n}\sum_{k=1}^{p}\left(\begin{array}{c}\dfrac{a_1^k.b_1^k+a_2^k.b_2^k+a_3^k.b_3^k+a_4^k.b_4^k}{\sqrt{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}.\sqrt{(b_1^k)^2+(b_2^k)^2+(b_3^k)^2+(b_4^k)^2}}.\\[2mm]\dfrac{w_q.\left[\min\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\min\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\min\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...\right]}{\max\left((\eta_A^1)_k,(\eta_B^1)_k\right)+\max\left((\eta_A^2)_k,(\eta_B^2)_k\right)+\max\left((\eta_A^3)_k,(\eta_B^3)_k\right)+...}\end{array}\right)\right)$$

$$= \lambda.\left(\frac{1}{p}\sum_{q=1}^{n}\sum_{k=1}^{p}\left(\left(1-\frac{\left|a_1^k-a_1^k\right|+\left|a_2^k-a_2^k\right|+\left|a_3^k-a_3^k\right|+\left|a_4^k-a_4^k\right|}{4}\right).\frac{w_q.\left[(\eta_A^1)_k.(\eta_A^1)_k+(\eta_A^2)_k.(\eta_A^2)_k+....+(\eta_A^p)_k.(\eta_A^p)_k\right]}{(\eta_A^1)_k^2+...+(\eta_A^p)_k^2+(\eta_A^1)_k^2+...+(\eta_A^p)_k^2-((\eta_A^1)_k^2+...(\eta_A^p)_k^2)}\right)\right)$$

$$+(1-\lambda).\left(\frac{1}{p}\sum_{q=1}^{n}\sum_{k=1}^{p}\left(\frac{\frac{a_1^k.a_1^k+a_2^k.a_2^k+a_3^k.a_3^k+a_4^k.a_4^k}{\sqrt{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}.\sqrt{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}}.}{\frac{w_q.\left[\min\left((\eta_A^1)_k,(\eta_A^1)_k\right)+\min\left((\eta_A^2)_k,(\eta_A^2)_k\right)+\min\left((\eta_A^3)_k,(\eta_A^3)_k\right)+...\right]}{\max\left((\eta_A^1)_k,(\eta_A^1)_k\right)+\max\left((\eta_A^2)_k,(\eta_A^2)_k\right)+\max\left((\eta_A^3)_k,(\eta_A^3)_k\right)+...}}\right)\right)$$

$$= \lambda.\left(\frac{1}{p}\sum_{q=1}^{n}\sum_{k=1}^{p}\left((1-0).\frac{w_q.\left[(\eta_A^1)_k^2+...+(\eta_A^p)_k^2\right]}{(\eta_A^1)_k^2+...+(\eta_A^p)_k^2}\right)\right)$$

$$+(1-\lambda).\left(\frac{1}{p}\sum_{q=1}^{n}\sum_{k=1}^{p}\left(\frac{\frac{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}{(a_1^k)^2+(a_2^k)^2+(a_3^k)^2+(a_4^k)^2}.}{\frac{w_q.\left[\min\left((\eta_A^1)_k,(\eta_A^1)_k\right)+\min\left((\eta_A^2)_k,(\eta_A^2)_k\right)+\min\left((\eta_A^3)_k,(\eta_A^3)_k\right)+...\right]}{\max\left((\eta_A^1)_k,(\eta_A^1)_k\right)+\max\left((\eta_A^2)_k,(\eta_A^2)_k\right)+\max\left((\eta_A^3)_k,(\eta_A^3)_k\right)+...}}\right)\right)$$

$$= \lambda.\sum_{q=1}^{n}w_q+(1-\lambda).\sum_{q=1}^{n}w_q$$

$$= 1.$$

5.Medical diagnosis using the Hybrid similarity measure

Fever, pain, weight change, fatigue, dizziness, cough, itching ... All these symptoms can have one or more meanings. Being aware of these symptoms plays a very important role in the early diagnosis and treatment of diseases. That is, that the individual is aware of the symptoms of the body has a great importance in the early diagnosis and treatment of illnesses

So, here we are to present an example of a medical diagnosis. Let $P =$ {Ali, Hasan, Ezgi} be a our set of patients. And let there be a set of diseases $D =$ {Measles, Cough, Flu} and let S={Backache, Stomachache, Earache} be a set of symptoms. Our solution is to examine the patient at different time intervals (four times a day). Let $\omega_1 = 0.6$, $\omega_2 = 0.4$.

Table 1: Q (the relation Between Patient and Symptoms)

Q	Backache	Stomachache	Earache
Ali	⟨(0.1,0.3,0.3,0.4); 0.1,0.3,0.6,0.7⟩	⟨(0.2,0.4,0.4,0.5); 0.2,0.4,0.7,0.8⟩	⟨(0.1,0.2,0.2,0.3); 0.2,0.4,0.6,0.8⟩
Hasan	⟨(0.2,0.4,04,0.5); 0.2,0.4,0.7,0.8⟩	⟨(0.1,0.2,0.2,0.3); 0.1,0.3,0.5,0.7⟩	⟨(0.2,0.4,0.4,0.7); 0.2,0.4,0.6,0.7⟩
Ezgi	⟨(0.1,0.2,0.3,0.4); 0.2,0.4,0.6,0.8⟩	⟨(0.4,0.6,0.6,0.7); 0.4,0.6,0.7,0.7⟩	⟨(0.3,0.4,0.7,0.8); 0.1,0.3,0.6,0.9⟩

Let us take the samples at four different timings in a day (in 08:30, 13:30, 18:30 and 23.30)

Table 2: R (the relation among Symptoms and Diseases)

R	Measles	Cough	Flu
Backache	⟨(0.1,0.2,0.4,0.4); 0.2,0.3,0.7,0.8⟩	⟨(0.1,0.2,0.5,0.6); 0.3,0.4,0.4,0.7⟩	⟨(0.1,0.2,0.3,0.5); 0.2,0.3,0.5,0.7⟩
Stomachache	⟨(0.2,0.4,0.5,0.6); 0.1,0.4,0.5,0.7⟩	⟨(0.2,0.2,0.5,0.7); 0.2,0.3,03,0.4⟩	⟨(0.3,0.4,0.5,0.6); 0.3,0.7,0.8,0.9⟩
Earache	⟨(0.5,0.6,0.7,0.8); 0.1,0.4,0.5,0.9⟩	⟨(0.1,0.2,0.5,0.6); 0.3,0.5,0.7,0.9⟩	⟨(0.3,0.5,0.6,0.8); 0.2,0.3,0.5,0.6⟩

Table 3: The Jaccard similarity measure Q and R

Jaccard	Measles	Cough	Flu
Ali	0,78719	**0,737372**	0,800518
Hasan	0,798176	**0,752882**	0,76456
Ezgi	**0,683783**	0,773332	0,728903
Optimal−Ali(Cough);Hasan(Cough); Ezgi(Measles)			

Table 4: The weighted Jaccard similarity measure Q and R

Weighted Jaccard	Measles	Cough	Flu
Ali	0,7871	**0,7373**	0,8005
Hasan	0,7981	**0,75287**	0,76455
Ezgi	**0,68378**	0,77332	0,72888
Optimal−Ali(Cough);Hasan(Cough); Ezgi(Measles)			

Table 5: The Cosine similarity measure Q and R.

Cosine	Measles	Cough	Flu
Ali	0,82508	**0,68566**	0,81249
Hasan	0,88142	**0,70496**	0,73462
Ezgi	**0,68887**	0,81786	0,71412
Optimal−Ali(Cough);Hasan(Cough); Ezgi(Measles)			

Table 6: The Weighted Cosine similarity measure Q and R

Weighted cosine	Measles	Cough	Flu
Ali	0,82507	**0,68566**	0,81248
Hasan	0,88141	**0,70495**	0,73641
Ezgi	**0,68886**	0,81786	0,71412
Optimal−Ali(Cough);Hasan(Cough); Ezgi(Measles)			

Table 7: The Hybrid Similarty measure Q and R with $\lambda = 0,9$ and $1-\lambda = 0,1$

Hybrid	Measles	Cough	Flu
Ali	0,790979	**0,732201**	0,801715
Hasan	0,8065	**0,74809**	0,761566
Ezgi	**0,684292**	0,777785	0,727425
Optimal−Ali(Cough);Hasan(Cough); Ezgi(Measles)			

Table 8: The Weighted Hybrid Similarty measure Q and R with $\lambda = 0,9$ and $1-\lambda = 0,1$

Weighted Hybrid	Measles	Cough	Flu
Ali	0,790897	**0,732136**	0,801698
Hasan	0,806431	**0,748078**	0,761736
Ezgi	**0,684288**	0,777774	0,727404
Optimal−Ali(Cough);Hasan(Cough); Ezgi(Measles)			

Table 9: Hybrid Similarity measure and Weighted Hybrid Similarity measure with optimal values

Similarity measure	Values	Measure value
$S_i = HybV(P,D)$	$\lambda = 0,9$	$HybV(Ali,Cough) = 0,732201$ $HybV(Hasan,Cough) = 0,74809$ $HybV(Ezgi,Measles) = 0,684292$
$S_i = HybV_{w_i}(P,D)$	$\lambda = 0,9$	$HybV_w(Ali,Cough) = 0,732136$ $HybV_w(Hasan,Cough) = 0,748078$ $HybV_w(Ezgi,Measles) = 0,684288$

6.Conclusions

In this chapter, a new hybrid similarity measure and a weighted hybrid similarity measure for trapezoidal fuzzy multi numbers are offered and some of its basic features are discussed. The suggested hybrid similarity measure strenghtenes the theories and techniques for measuring the degree of hybrid similarity. This measure widely desreases the influence of uncertain measures and ensures an highly intuitive quantification. The

effectiveness and efficiency of the proposed hybrid similarity measure is verified in a numerical example with the help of measure of error and measure of performance. Furthermore, medical diagnosis problems have been displayed through a hypothetical case study by using this proposed hybrid similarity measure. The authors hope that the suggested idea can be performed in solving realistic multi-criteria decision making problems.

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Chapter Thirteen

Dice Vector Similarity Measure of Trapezoidal Fuzzy Multi-Numbers Based On Multi-Criteria Decision Making

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ABSTRACT
The fundamental purpose of this chapter is to introduce a novel approach based on multi-criteria decision making(MCDM) trapezoidal fuzzy multi-number. Therefore, Dice vector similarity and weighted Dice vector similarity measure is defined to develop the Trapezoidal Fuzzy Multi-Numbers. In addition, the method is applied to a numerical example one may supposing to confirm the practicality and certainity of the submitted approach.

Keywords: Neutrosophic set, Trapezoidal Fuzzy Multi-Numbers, Dice vector similarity measure, MCDM.

1.Introduction

In 1965, Zadeh [1] came up with the concept of fuzzy sets which includes claasical universe set A to process defective, obscure, suspicious and indefinite information of fuzzy sets. Then, the description of fuzzy set has been conveniently carried out in science and technology, artificial intellegence, multifactor systems, computational modelling ,etc. A membership value of fuzzy sets are [0,1] in a universe; but, it is insufficient for supplying exact result of some problems because it may have status with distinct membership values for each member. Therefore, a distinct combination of fuzzy sets, that is, it was suggested the concept of multi-fuzzy sets. Yager [2] initially offered multi-fuzzy sets as a combination of fuzzy sets with multisets. There may be more than one membership value in [0, 1] which is a member of a multi-fuzzy set (that is, there may be recurring cases of an element). Miyamoto [3, 4], Maturo [5], Sebastian and Ramakrishnan [6], Syropoulos [7, 8] and others run several works on the multi-sets. Lately, intensive research has been made on fuzzy numbers. For instance, several arithmetic processes with linear membership functions on fuzzy numbers are improved by Thowhida and Ahmad [9]. Chakrabort and Guha [10] and Alim et al [11] improved several arithmetic operations and a method for the basic operations on generalized fuzzy numbers and L-R fuzzy number by utilizing extension basis. Roseline and Amirtharaj [12] advanced generalized fuzzy Hungarian method and approved a formula of estimating of generalized trapezoidal fuzzy numbers. In addition, similar researchers in [13] proposed a technique of estimating of generalized

trapezoidal fuzzy numbers depend on rank, perimeter, etc. Meng et al [14] analysed a multi-attribute decision-making problem along attribute values in triangular fuzzy numbers. Surapati and Biswas [15] viewed a multiple objective assignment problem along uncertain costs, time and ineffectiveness in place of its exact data in fuzzy numbers. Sinova et al. [16] put forward a classification of the handling of several haphazard factors by expanding the occasion-developing function in fuzzy numbers. Riera and Torrens [17] improved a way on discrete fuzzy numbers to pattern real and unreal qualitative data. There has been several studies researched with fuzzy numbers recently. For instance; on existence, singleness, calculus and features of triangular approachings of fuzzy numbers [18], t-norms and s-norms with two valued functions t and s which transform from [0, 1] * [0, 1] into [0, 1], [19], data system and operations for fuzzy multi-sets [20], optimization by interval and fuzzy numbers [26], selecting them based on variance [27], defuzzification of fuzzy numbers and its application in many areas can be revealed in [28], [29], and [30]. Finally, the application of NET to algebraic structures and the similarity among two different algebraic systems can be seen in [31] and [32].

2. Preliminary

This section reviews some basic facts on the fuzzy set, fuzzy number and multi-fuzzy set.

2.1 Fuzzy Sets [1]

Let X be a non-empty set. A fuzzy set F on X is defined as:

$$F = \{\langle x, \mu_F(\mathrm{x})\rangle : \mathrm{x} \in \mathrm{X}\} \text{ where } \mu_F : X \to [0,1] \text{ for } \mathrm{x} \in X .$$

2.2 Multi-fuzzy SetS [6]

Let X be a non-empty set. A multi-fuzzy set G on X is defined as $G = \left\{\langle x, \mu_G^1(\mathrm{x}), \mu_G^2(\mathrm{x}),...,\mu_G^i(\mathrm{x}),...\rangle : x \in X\right\}$ where $\mu_G^i : X \to [0,1]$ for all $\mathrm{i} \in \{1,2,...,p\}$ and $x \in X$

2.3 Trapezoidal Fuzzy Multi-number [21]

Let $\eta_A^i \in [0,1]$ $(\mathrm{i} \in \{1,2,...,p\})$ and $a,b,c,d \in \mathbb{R}$ such that $a \le b \le c \le d$. Then, a (TFM-number) $\tilde{a} = \langle(a,b,c,d);\eta_A^1,\eta_A^2,...,\eta_A^p\rangle$ is a special fuzzy multi-set on the real number set $\mathbb{R}$, whose membership functions are defined as

$$\mu_A^i(x) = \begin{cases} (x-a_1)\eta_A^i/(b_1-a_1) & a_1 \le x \le b_1 \\ \eta_A^i & b_1 \le x \le c_1 \\ (d_1-x)\eta_A^i/(d_1-c_1) & c_1 \le x \le d_1 \\ 0 & otherwise \end{cases}$$

Note : The set of all TFM-number on $\mathbb{R}$ is denoted by Λ .

2.4 Some Operation on TFM-number [21]

Let $A = \left\langle (a_1,b_1,c_1,d_1);\eta_A^1,\eta_A^2,...,\eta_A^p \right\rangle$, $B = \left\langle (a_2,b_2,c_2,d_2);\eta_B^1,\eta_B^2,...,\eta_B^p \right\rangle \in \Lambda$ and $\gamma \ne 0$ be any real number. Then,

1. $A+B = \left\langle (a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2);s(\eta_A^1,\eta_B^1),s(\eta_A^2,\eta_B^2),...,s(\eta_A^P,\eta_B^P) \right\rangle$

2. $A-B = \left\langle (a_1-d_2,b_1-c_2,c_1-b_2,d_1-a_2);s(\eta_A^1,\eta_B^1),s(\eta_A^2,\eta_B^2),...,s(\eta_A^P,\eta_B^P) \right\rangle$

3. $A.B = \begin{cases} \left\langle (a_1a_2,b_1b_2,c_1c_2,d_1d_2); \\ t(\eta_A^1,\eta_B^1),t(\eta_A^2,\eta_B^2),...,t(\eta_A^p,\eta_B^p) \right\rangle & (d_1>0,d_2>0) \\ \left\langle (a_1d_2,b_1c_2,c_1b_2,d_1a_2); \\ t(\eta_A^1,\eta_B^1),t(\eta_A^2,\eta_B^2),...,t(\eta_A^p,\eta_B^p) \right\rangle & (d_1<0,d_2>0) \\ \left\langle (d_1d_2,c_1c_2,b_1b_2,a_1a_2); \\ t(\eta_A^1,\eta_B^1),t(\eta_A^2,\eta_B^2),...,t(\eta_A^p,\eta_B^p) \right\rangle & (d_1<0,d_2<0) \end{cases}$

4. $A/B = \begin{cases} \left\langle (a_1/d_2,b_1/c_2,c_1/b_2,d_1/a_2); \\ t(\eta_A^1,\eta_B^1),t(\eta_A^2,\eta_B^2),...,t(\eta_A^p,\eta_B^p) \right\rangle & (d_1>0,d_2>0) \\ \left\langle (d_1/d_2,c_1/c_2,b_1/b_2,a_1/a_2); \\ t(\eta_A^1,\eta_B^1),t(\eta_A^2,\eta_B^2),...,t(\eta_A^p,\eta_B^p) \right\rangle & (d_1<0,d_2>0) \\ \left\langle (d_1/a_2,c_1/b_2,b_1/c_2,a_1/d_2); \\ t(\eta_A^1,\eta_B^1),t(\eta_A^2,\eta_B^2),...,t(\eta_A^p,\eta_B^p) \right\rangle & (d_1<0,d_2<0) \end{cases}$

5. $\gamma A = \left\langle (\gamma a_1,\gamma b_1,\gamma c_1,\gamma d_1);1-(1-\eta_A^1)^\gamma,1-(1-\eta_A^2)^\gamma,...,1-(1-\eta_A^p)^\gamma \right\rangle (\gamma \ge 0)$

6. $A^\gamma = \left\langle (a_1^\gamma,b_1^\gamma,c_1^\gamma,d_1^\gamma);(\eta_A^1)^\gamma,(\eta_A^2)^\gamma,...,(\eta_A^P)^\gamma \right\rangle (\gamma \ge 0)$

2.5 Normalized TFM-number [21]

Let $A = \left\langle (a_1, b_1, c_1, d_1) : \eta_A^1, \eta_A^2, ..., \eta_A^p \right\rangle \in \Lambda$. Then,

$$\bar{A} = \left\langle \left(\frac{a_1}{a_1 + b_1 + c_1 + d_1}, \frac{b_1}{a_1 + b_1 + c_1 + d_1}, \frac{c_1}{a_1 + b_1 + c_1 + d_1}, \frac{d_1}{a_1 + b_1 + c_1 + d_1} \right); \eta_A^1, \eta_A^2, ..., \eta_A^p \right\rangle \text{ is a}$$

normalized TFM-number of A

Numerical example: Assume that $A = \left\langle (1, 3, 5, 6); 0.3, 0.2, ..., 0.8 \right\rangle \in \Lambda$. Here :

$$\bar{A} = \left\langle \left(\frac{1}{15}, \frac{3}{15}, \frac{5}{15}, \frac{6}{15} \right); 0.3, 0.2, ..., 0.8 \right\rangle \text{ is a normalized TFM-number of A .}$$

2.6 Dice similarity measure [22]

Let $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ be the two vectors of length n where all the coordinates are positive. Then the DSM between two vectors are given as follows:

$$D(X, Y) = \frac{2X.Y}{[\![X]\!]_2^2 + [\![Y]\!]_2^2} = \frac{2\sum\limits_{i=1}^{n} x_i y_i}{\sum\limits_{i=1}^{n} x_i^2 + \sum\limits_{i=1}^{n} y_i^2} \qquad (14)$$

where $X.Y = \sum\limits_{i=1}^{n} x_i y_i$ is the inner product of the vectors X and Y and $[\![X]\!]_2 = \sqrt{\sum\limits_{i=1}^{n} x^2}$ and

$[\![Y]\!]_2 = \sqrt{\sum\limits_{i=1}^{n} y^2}$ are the Euclidean norms of X and Y (L_2 norms). However, it is undefined if

$x_i = y_i = 0$ for $i = 1, 2, ..., n$. In this case, let the measure value be zero when $x_i = y_i = 0$ for $i = 1, 2, ..., n$.

The DSM satisfies the following properties (22; 24,25:)

$$(\text{P1}) \ 0 \leq D(X, Y) \leq 1;$$

$$(\text{P2}) \ D(X, Y) = D(Y, X)$$

$$(\text{P3}) \ D(X, Y) = 1 \text{ if and only if } X = Y \text{ ,i.e. } x_i = y_i, \text{ for i=1,2,...,n.}$$

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The DSM in vector space can be extended to the following expected DSM for trapezoidal fuzzy numbers.

3. Dice Vector Similarity Measure Depend on MCDM with TFM -Numbers)

3.1 DSM between TFM-number R and S

Let $R = \left\langle (r_1, r_2, r_3, r_4); \eta_R^1, \eta_R^2, ..., \eta_R^p \right\rangle$, $S = \left\langle (s_1, s_2, s_3, s_4); \eta_S^1, \eta_S^2, ..., \eta_S^p \right\rangle$ be two TFMNs in the set of real numbers $\mathbb{R}$. Then; DSM between TFMN R and S denoted $D(R, S)$ is defined as;

$$D(R,S) = \frac{1}{n} \sum_{j=1}^{n} \left(\frac{1}{1+d(R,S)} \cdot \frac{2(\eta_R^1(x_j).\eta_S^1(x_j) + \eta_R^2(x_j).\eta_S^2(x_j) + ... + \eta_R^p(x_j).\eta_S^p(x_j))}{(\eta_R^1)^2(x_j) + (\eta_S^1)^2(x_j) + (\eta_R^2)^2(x_j) + (\eta_S^2)^2(x_j) + ... + (\eta_R^P)^2(x_j) + (\eta_S^P)^2(x_j)} \right)$$

$$d(R,S) = |P(R) - P(S)|$$

$$P(R) = \frac{r_1 + 2r_2 + 2r_3 + r_4}{6} , \quad P(S) = \frac{s_1 + 2s_2 + 2s_3 + s_4}{6}$$

Proposition 3.1 Let $D(R, S)$ be a DSM between TFMN's R and S. Then we have,

i. $0 \le D(R,S) \le 1$

ii. $D(R,S) = D(S,R)$

iii. $D(R,S) = 1$ *for* $R = S$

Numerical Example

Let $R = \left\langle (1,2,3,4); 0.3, 0.2, 0.4, 0.6 \right\rangle$, $S = \left\langle (3,5,7,9); 0.2, 0.3, 0.4, 0.5 \right\rangle$ be two TFMNs in the set of real numbers $\mathbb{R}$. Then; DSM between TFMN R and S

$$d(R,S) = |P(R) - P(S)| = |2,5 - 6| = 3,5$$

$$P(R) = \frac{1+4+6+4}{6} = \frac{15}{6} = \frac{5}{2} = 2,5$$

$$P(S) = \frac{3+10+14+9}{6} = \frac{36}{6} = 6$$

$$D(R,S) =$$

$$\frac{1}{4}\sum_{j=1}^{n}\left(\frac{1}{1+3,5}\cdot\frac{2(0,3.0,2+0,2.0,3+0,4.0,4+0,6.0,5)}{(0,3)^2+(0,2)^2+(0,2)^2+(0,3)^2+(0,4)^2+(0,4)^2+(0,6)^2+(0,5)^2}\right)$$

$$=\frac{1}{4}\cdot\frac{1}{1+3,5}\cdot\frac{2.0,58}{1,19}=\frac{1,16}{4.4,5.1,19}=\frac{1,16}{21,42}\cong 0.054$$

3.3 DSM within TFM-numbers

Let $R=\left\langle(r_1,r_2,r_3,r_4);\eta_R^1,\eta_R^2,...,\eta_R^p\right\rangle$, $S=\left\langle(s_1,s_2,s_3,s_4);\eta_S^1,\eta_S^2,...,\eta_S^p\right\rangle$ be two TFMNs in

the set of real numbers $\mathbb{R}$ and $w_i\in[0,1]$ be the weight of each element x_i for $i=(1,2,...,n)$

so as to $\sum_{i=1}^{n}w_i=1$. Subsequently; DSM within TFMN R and S denoted $D_w(R,S)$ is defined

as;

$$D_w(R,S)=\sum_{i=1}^{n}\left(\frac{1}{1+d(R,S)}\cdot\frac{2w_i(\eta_R^1(x_j).\eta_S^1(x_j)+\eta_R^2(x_j).\eta_S^2(x_j)+....+\eta_R^p(x_j)\eta_S^p(x_j))}{(\eta_R^1(x_j))^2+(\eta_S^1(x_j))^2+(\eta_R^2(x_j))^2+(\eta_S^2(x_j))^2+...+(\eta_R^P(x_j))^2+(\eta_S^P(x_j))^2}\right)$$

$$d(R,S)=\left|P(R)-P(S)\right|$$

$$P(R)=\frac{r_1+2r_2+2r_3+r_4}{6},\quad P(S)=\frac{s_1+2s_2+2s_3+s_4}{6}$$

Proposition 3.2 Let $D_w(\bar{R},\bar{S})$ be a weighted DSM between normalized TFMN's R and S,

$w_j\in[0,1]$ be the weight of each element x_j such that $\sum_{j=1}^{n}w_j=1$ Then weighted Dice

vector similarity measure between TFMN's R and S;

i. $0\le D_w(\bar{R},\bar{S})\le 1$

ii. $D_w(\bar{R},\bar{S})=D_w(\bar{S},\bar{R})$

iii. $D_w(\bar{R},\bar{S})=1$ *for* $\bar{R}=\bar{S}$ i.e. ($\eta_R^1=\eta_S^1,\eta_R^2=\eta_S^2,...,\eta_R^p=\eta_S^p$)

Proof

i. It is clear from 3.3

ii.

$$D_w(\bar{R},\bar{S}) = \sum_{j=1}^{n}\left(\frac{1}{1+\left|\frac{r_1+2s_1+2t_1+u_1}{6}-\frac{r_2+2s_2+2t_2+u_2}{6}\right|} \cdot \frac{2w_j(\eta_{\bar{R}}^1(x_j).\eta_{\bar{S}}^1(x_j)+\eta_{\bar{R}}^2(x_j).\eta_{\bar{S}}^2(x_j)+....+\eta_{\bar{R}}^p(x_j)\eta_{\bar{S}}^p(x_j))}{(\eta_{\bar{R}}^1)^2(x_j)+(\eta_{\bar{S}}^1)^2(x_j)+(\eta_{\bar{R}}^2)^2(x_j)+(\eta_{\bar{S}}^2)^2(x_j)+...+(\eta_{\bar{R}}^P)^2(x_j)+(\eta_{\bar{R}}^P)^2(x_j)} \right)$$

$$= D_w(\bar{R},\bar{S}) = \sum_{j=1}^{n}\left(\frac{1}{1+\left|\frac{r_2+2s_2+2t_2+u_2}{6}-\frac{r_1+2s_1+2t_1+u_1}{6}\right|} \cdot \frac{2w_j(\eta_{\bar{S}}^1(x_j).\eta_{\bar{R}}^1(x_j)+\eta_{\bar{S}}^2(x_j).\eta_{\bar{R}}^2(x_j)+....+\eta_{\bar{S}}^p(x_j).\eta_{\bar{R}}^p(x_j))}{(\eta_{\bar{S}}^1)^2(x_j)+(\eta_{\bar{R}}^1)^2(x_j)+(\eta_{\bar{S}}^2)^2(x_j)+(\eta_{\bar{R}}^2)^2(x_j)+...+(\eta_{\bar{S}}^P)^2(x_j)+(\eta_{\bar{R}}^P)^2(x_j)} \right)$$

$$= D_w(\bar{S},\bar{R})$$

iii.

$$D_w(\bar{S},\bar{R})$$

$$= \sum_{j=1}^{n}\left(\frac{1}{1+\left|\frac{r_1+2s_1+2t_1+u_1}{6}-\frac{r_2+2s_2+2t_2+u_2}{6}\right|} \cdot \frac{2w_j(\eta_{\bar{R}}^1(x_j).\eta_{\bar{S}}^1(x_j)+\eta_{\bar{R}}^2(x_j).\eta_{\bar{S}}^2(x_j)+....+\eta_{\bar{R}}^p(x_j)\eta_{\bar{S}}^p(x_j))}{(\eta_{\bar{R}}^1)^2(x_j)+(\eta_{\bar{S}}^1)^2(x_j)+(\eta_{\bar{R}}^2)^2(x_j)+(\eta_{\bar{S}}^2)^2(x_j)+...+(\eta_{\bar{R}}^P)^2(x_j)+(\eta_{\bar{S}}^P)^2(x_j)} \right)$$

$$= \sum_{j=1}^{n}\left(\frac{1}{1+\left|\frac{r_1+2s_1+2t_1+u_1}{6}-\frac{r_1+2s_1+2t_1+u_1}{6}\right|} \cdot \frac{2w_j(\eta_{\bar{R}}^1(x_j).\eta_{\bar{R}}^1(x_j)+\eta_{\bar{R}}^2(x_j).\eta_{\bar{R}}^2(x_j)+....+\eta_{\bar{R}}^p(x_j)\eta_{\bar{R}}^p(x_j))}{(\eta_{\bar{R}}^1)^2(x_j)+(\eta_{\bar{R}}^1)^2(x_j)+(\eta_{\bar{R}}^2)^2(x_j)+(\eta_{\bar{R}}^2)^2(x_j)+...+(\eta_{\bar{R}}^P)^2(x_j)+(\eta_{\bar{R}}^P)^2(x_j)} \right)$$

$$= \sum_{j=1}^{n}\left(\frac{1}{1+0} \cdot \frac{2w_j\left((\eta_{\bar{R}}^1)^2(x_j)+(\eta_{\bar{R}}^2)^2(x_j)+....+(\eta_{\bar{R}}^p)^2(x_j)\right)}{2\left((\eta_{\bar{R}}^1)^2(x_j)+(\eta_{\bar{R}}^2)^2(x_j)+....+(\eta_{\bar{R}}^p)^2(x_j)\right)} \right)$$

$$=1.$$

Numerical Example

Let $R = \langle (2,3,5,6); 0.2,0.5,0.6,0.9\rangle$, $S = \langle (1,2,4,5); 0.3,0.4,0.5,0.7\rangle$ be two TFMNs in the set of real numbers $\mathbb{R}$ and w_i be the weight of each element x_i for $i = (1,2)$ $w_1 = 0.6, w_2 = 0.4$ such that $\sum_{i=1}^{n} w_i = 1$. Then; Dice similarity measure between TFMN R and S is;

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$$d(R,S) = |P(R) - P(S)| = |4 - 3| = 1$$

$$P(R) = \frac{2+6+10+6}{6} = \frac{24}{6} = 4$$

$$P(S) = \frac{1+4+8+5}{6} = \frac{18}{6} = 3$$

$$D_w(R,S) = \frac{1}{(1+1)} \cdot \frac{2.(0,6).(0,2.0,3+0,5.0,4+0,6.0,5+0,9.0,7)}{(0,2)^2 + (0,3)^2 + (0,5)^2 + (0,4)^2 + (0,6)^2 + (0,5)^2 + (0,9)^2 + (0,7)^2}$$

$$+ \frac{1}{(1+1)} \cdot \frac{2.(0,4).(0,2.0,3+0,5.0,4+0,6.0,5+0,9.0,7)}{(0,2)^2 + (0,3)^2 + (0,5)^2 + (0,4)^2 + (0,6)^2 + (0,5)^2 + (0,9)^2 + (0,7)^2}$$

$$= \frac{1}{2} \cdot \frac{2.(0,6).(1,19)}{2,45} + \frac{1}{2} \cdot \frac{2.(0,4).(1,19)}{2,45}$$

$$= 0,48$$

4. TFM-number MCDM Method

In this section, we define TFMN and MCDM method depend on Dice vector similarity measure for TFM-numbers.

4.1 TFM-number MCDM Matrix

Assume that $U = (u_1, u_2, \ldots, u_m)$ be a set of alternatives, $R = (r_1, r_2, \ldots, r_n)$ be the set of criteria, $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of the $r_j (j = 1,2 \ldots, n)$ such that $w_j \geq 0$ and $\sum_{i=1}^{n} w_i = 1$ and $[b_{ij}]_{mxn} = \langle (ar_{ij}, b_{ij}, c_{ij}, d_{ij}); \eta_{ij}^1, \eta_{ij}^2, \ldots, \eta_{ij}^p \rangle$ be the decision matrix whither the ranking values of the options. Then

$$\left[b_{ij} \right]_{m \times n} = \begin{matrix} & \begin{matrix} a_1 & a_2 & \cdots & a_n \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{matrix} & \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \end{matrix}$$

is called a TFM-number MCDM matrix of the decision maker.

Also; r^+ is positive ideal TFM-numbers solution of decision matrix $[b_{ij}]_{m \times n}$ as form:

$$r^+ = \langle(1,1,1,1); 1,1,\dots,1\rangle$$

and r^- is negative ideal TFM-numbers solution of decision matrix $[b_{ij}]_{m\times n}$ as form:

$$r^- = \langle(0,0,0,0); 0,0,\dots,0\rangle.$$

Algorithm:

Step1. Give the decision-making matrix $[b_{ij}]_{m\times n}$; for decision;

Step2. Calculate the weightedDice vector similarity S_i between positive ideal (or negative ideal) TFMN solution r^+ and $u_i = \langle(a_i, b_i, c_i, d_i); \eta_A^1, \eta_A^2, \dots, \eta_A^p\rangle$ and $(i = 1,2\dots,m)$ as;

$$D_w(R,S) = \sum_{j=1}^{n}\left(\frac{1}{1+d(R,S)}\cdot\frac{2w_j(\eta_R^1(x_j).\eta_S^1(x_j)+\eta_R^2(x_j).\eta_S^2(x_j)+\dots+\eta_R^p(x_j)\eta_S^p(x_j))}{(\eta_R^1(x_j))^2+(\eta_S^1(x_j))^2+(\eta_R^2(x_j))^2+(\eta_S^2(x_j))^2+\dots+(\eta_R^P(x_j))^2+(\eta_S^P(x_j))^2}\right)$$

Step3. Determine the non-increasing order of $S_i = D_{w_i}(u_i, r^+)\ (i = 1,2,\dots,m)$, $(j = 1,2,\dots,n)$

Step4. Select the best option.

Let's see the following numerical example;

Numerical Example

Let's consider decision making problem adapted from Xu and Cia [23]. We consider Nizip Medical who intends to stretcher. Four types of stretchers (alternatives) $u_i(i = 1,2,3,4)$ are able to be used.. The customer takes into account four attributes to evaluate the alternatives; a_1 =collapsible stretcher; a_2 =rollerstretcher; a_3 =hammock stretcherand use the TFMN values to calculate the four possible options $u_i(i = 1, 2, 3, 4)$ according to the above four attributes. Also, the weight vector of the attributes $a_j(j = 1,2,3,4)$ is $\omega = (0.2,0.5,0.1,0.2)^T$. Then,

Algorithm

Step1. Constructed the decision matrix supplied by the Nizip Medical as:

	r_1	r_2	r_3
u_1	$\langle(0.3, 0.5, 0.7, 0.9); 0.4, 0.5, 0.3, 0.6\rangle$	$\langle(0.6, 0.7, 0.8, 0.9); 0.8, 0.9, 0.6, 0.3\rangle$	$\langle(0.1, 0.3, 0.5, 0.8); 0.2, 0.5, 0.2, 0.1\rangle$
u_2	$\langle(0.2, 0.3, 0.4, 0.5); 0.8, 0.1, 0.4, 0.2\rangle$	$\langle(0.5, 0.6, 0.8, 0.9); 0.1, 0.9, 0.3, 0.7\rangle$	$\langle(0.2, 0.5, 0.8, 0.9); 0.7, 0.7, 0.1, 0.3\rangle$
u_3	$\langle(0.1, 0.5, 0.6, 0.7); 0.2, 0.6, 0.2, 0.5\rangle$	$\langle(0.4, 0.6, 0.7, 0.9); 0.2, 0.9, 0.1, 0.8\rangle$	$\langle(0.5, 0.6, 0.7, 0.8); 0.8, 0.8, 0.5, 0.1\rangle$
u_4	$\langle(0.3, 0.4, 0.6, 0.8); 0.6, 0.9, 0.1, 0.2\rangle$	$\langle(0.2, 0.3, 0.7, 0.8); 0.8, 0.3, 0.2, 0.4\rangle$	$\langle(0.1, 0.5, 0.6, 0.8); 0.2, 0.3, 0.1, 0.3\rangle$

Table 3: Decision matrix stated by Nizip Medical

Step2. Computed the positive ideal TFM-numbers solution as;

$$r^+ = \langle(1, 1, 1, 1); 1, 1, \dots, 1\rangle$$

Step3. Calculated the weighted Dice vector similarity measures, $S_i = D_{w_i}(u_i, r^+)$ as;

The Proposed method	Measure value	Ranking order
$S_i = D_{w_i}(u_i, r^+)$	$D_{w_i}(u_1, r^+) = 0,1302$ $D_{w_i}(u_2, r^+) = 0,3341$ $D_{w_i}(u_3, r^+) = 0,0685$ $D_{w_i}(u_4, r^+) = 0,1197$	$S_2 > S_1 > S_4 > S_3$

Step 4. So the Medical will choose the stretcher u_2. Nevertheless if they don't select u_2 as a result of a few causes an alternative choice will be u_1.

6. Conclusions

In this chapter, we developed a MCDM for trapezoidal fuzzy multi-number based on weighted Dice vector similarity measures and applied to a numerical example in order to confirm the practicality and accuracy of the proposed method. In the future, the method

can be extend with different similarity and distance measures in intuitionistic fuzzy set and neutrosophic set.

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The **Neutrosophic Triplets** were introduced by F. Smarandache & M. Ali in 2014 – 2016, and consequently the neutrosophic triplet group, ring, field - in general the neutrosophic triplet structures; while the **Neutrosophic Extended Triplets** were introduced by F. Smarandache in 2016 and consequently the neutrosophic extended triplet structures:

http://fs.unm.edu/NeutrosophicTriplets.htm

<u>Definition of Neutrosophic Triplet (NT).</u>

A neutrosophic triplet is an object of the form $<x, neut(x), anti(x)>$, for $x \in N$, where

$neut(x) \in N$ is the neutral of x, different from the classical algebraic unitary element if any, such that:

$$x*neut(x) = neut(x)*x = x$$

and $anti(x) \in N$ is the opposite of x such that:

$$x*anti(x) = anti(x)*x = neut(x).$$

In general, an element x may have more *anti*'s.

<u>Definition of Neutrosophic Extended Triplet (NET).</u>

A neutrosophic extended triplet is a neutrosophic triplet, defined as above, but where the *neutral* of x {denoted by $_eneut(x)$ and called "extended neutral"} is allowed to also be equal to the classical algebraic unitary element (if any). Therefore, the restriction "different from the classical algebraic unitary element if any" is released.

As a consequence, the "extended opposite" of x, denoted by $_eanti(x)$, is also allowed to be equal to the classical inverse element from a classical group.

Thus, a neutrosophic extended triplet is an object of the form $<x, _eneut(x), _eanti(x)>$, for $x \in N$, where $_eneut(x) \in N$ is the *extended neutral* of x, which can be equal or different from the classical algebraic unitary element if any, such that:

$$x*_eneut(x) = _eneut(x)*x = x$$

and $_eanti(x) \in N$ is the *extended opposite* of x such that:

$$x*_eanti(x) = _eanti(x)*x = _eneut(x).$$

In general, for each $x \in N$ there are may exist many $_eneut$'s and $_eanti$'s.